De Moivres Theorem

$(\cos x + i \sin x)^n = \cos nx + i \sin nx$

De Moivre's theorem If z = r(cos α + i sin α), and n is a natural number, then

- $z^n = r^n(\cos n\alpha + i \sin n\alpha)$

Examples

Question 1: Solve $(\cos(\frac{-\pi}{4}) + i \sin(\frac{-\pi}{4}))^{10}$

Solution:

Let $Z = (\cos(\frac{-\pi}{4}) + i \sin(\frac{-\pi}{4}))^{10}$

According to De moiver Theorem:

$(\cos\theta + i \sin \theta)n = \cos n\theta + i \sin n\theta$

$\Rightarrow Z = \cos(\frac{-10\pi}{4}) + i \sin(\frac{-10\pi}{4})$

$= \cos(\frac{-5\pi}{2}) + i \sin(\frac{-5\pi}{2})$

$= \cos(2\pi + \frac{-5\pi}{2}) + i \sin(2\pi + \frac{-5\pi}{2})$

$= \cos(\frac{-\pi}{2}) + i \sin(\frac{-\pi}{2})$

$= 0 + i(-1)$

$= -i$

Question 2: If Z = cos 30^o + i sin 30^o, find Z^7.

Solution:

Given Z = cos 30^0 + i sin 30^0

Now, Z^7 = (cos 30^0 + i sin 30^0)^7

According to De moiver Theorem:

(Cosθ + i sin θ)n = Cos nθ + i sin nθ

=> Z^7 = cos (7 * 30^0) + i sin (7 * 30^0)

= cos 210^0 + i sin 210^0

= cos (π + 30^o) + i sin (π + 30^o)

[In 3rd quadrant both the given functions are -ve]

= - cos 30^0 - i sin 30^0

= - $\frac{\sqrt{3}}{2}$ - $\frac{1}{2}$ i

= - $\frac{1}{2}$ ($\sqrt{3}$ + i)

Examples

Example 1: Write $\left(\sqrt{3}+i\right)^7$ in the form *s + bi.*

First determine the radius:

$$r=\left|\sqrt{3}+i\right|$$
$$r=\sqrt{\sqrt{3}^2+1^2}$$
$$r=\sqrt{3+1}$$
$$r=2$$

Since cos α = $\sqrt{3}/2$ and sin α = ½, α must be in the first quadrant and α = 30°. Therefore,

$$\left(\sqrt{3}+i\right)^7=\left[2\left(\cos 30°+i\sin 30°\right)\right]^7$$
$$\left(\sqrt{3}+i\right)^7=2^7\left[\cos\left(7\cdot 30°\right)+i\sin\left(7\cdot 30°\right)\right]$$
$$\left(\sqrt{3}+i\right)^7=128\left(\cos 210°+i\sin 210°\right)$$
$$\left(\sqrt{3}+i\right)^7=128\left(-\frac{\sqrt{3}}{2}+\frac{-1}{2}i\right)$$
$$\left(\sqrt{3}+i\right)^7=128\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$$
$$\left(\sqrt{3}+i\right)^7=-64\sqrt{3}-64i$$

Example 2: Write $\left(\sqrt{2} - i\sqrt{2}\right)^4$ in the form *a + bi*.

First determine the radius:

$$r = \left|\sqrt{2} - i\sqrt{2}\right|$$
$$r = \sqrt{\sqrt{2}^2 + \left(-\sqrt{2}\right)^2}$$
$$r = \sqrt{2 + 2}$$
$$r = 2$$

Since cos α = !α = $\sqrt{2}/2$ and sin $\alpha = -\sqrt{2}/2$, α must be in the fourth quadrant and α = 315°. Therefore,

$$\left(\sqrt{2} - i\sqrt{2}\right)^4 = \left[2\left(\cos 315^\circ + i\sin 315^\circ\right)\right]^4$$
$$\left(\sqrt{2} - i\sqrt{2}\right)^4 = 2^4\left[\cos\left(4 \cdot 315^\circ\right) + i\sin\left(4 \cdot 315^\circ\right)\right]$$
$$\left(\sqrt{2} - i\sqrt{2}\right)^4 = 16\left(\cos 1260^\circ + i\sin 1260^\circ\right)$$
$$\left(\sqrt{2} - i\sqrt{2}\right)^4 = 16\left(\cos 180^\circ + i\sin 180^\circ\right)$$
$$\left(\sqrt{2} - i\sqrt{2}\right)^4 = 16(-1 + 0i)$$
$$\left(\sqrt{2} - i\sqrt{2}\right)^4 = -16 + 0i$$
$$\left(\sqrt{2} - i\sqrt{2}\right)^4 = -16$$

Exercises

1 Find $(1 - i)^6$

2 Find $(\sqrt{3} - i)^9$