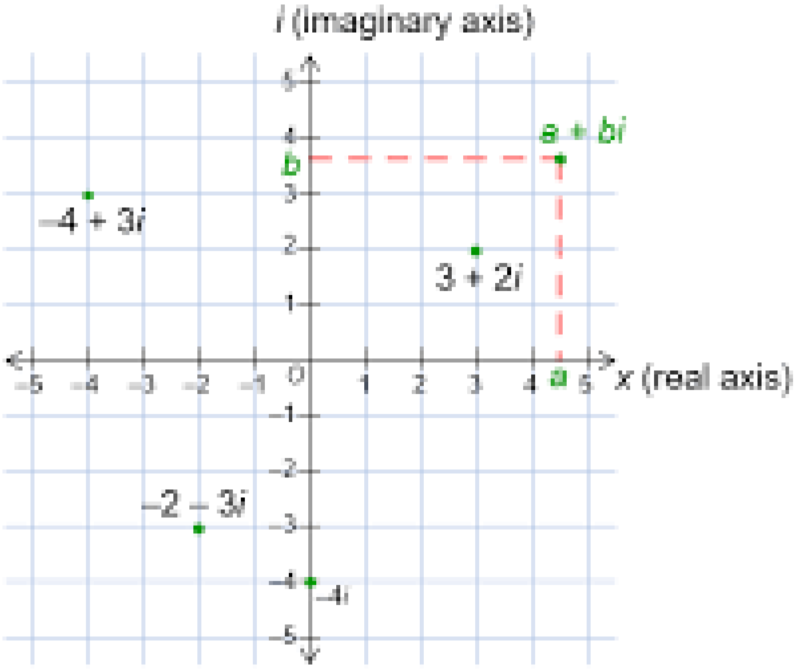
Argand Diagram



Addition and Subtraction

$$(5+2i)+(3-2i)=$$

$$(-4+2i)+(-1-2i)=$$

$$(5+2i)-(3-2i)=$$

$$(2+2i)+(-3-2i)=$$

Multiplication and Division

Questions

1) $(8+7i)^2$

2) $(3+8i)(-2-i)$

3) $(-1+6i)^2$

4) $(-1-8i)(6-5i)$

5) $(8-2i)^2$

6) $(8+5i)(-6-2i)$

7) $8(-2i)(-5-4i)$

8) $(2-2i)^2$

9) $(4i)(-2i)(-8+4i)$

10) $(-4-4i)(1-3i)$

11) $(7-2i)(6-4i)$

12) $(4-3i)(6-6i)$

13) $\frac{1+9i}{-2+9i}$

14) $\frac{-7+6i}{7-10i}$

15) $\frac{-5-i}{-6-9i}$

16) $\frac{4-8i}{6-5i}$

17) $\frac{-1+3i}{-4-8i}$

18) $\frac{-4+4i}{4-6i}$

19) $\frac{-6-i}{1+5i}$

20) $\frac{-4+3i}{-10+7i}$

Answers

1) $15+112i$
2) $2-19i$
3) $-35-12i$
4) $-46-43i$
5) $60-32i$
6) $-38-46i$
7) $-64+80i$
8) $-8i$
9) $-64+32i$
10) $-16+8i$
11) $34-40i$
12) $6-42i$
13) $\frac{79}{85}-\frac{27i}{85}$
14) $-\frac{109}{149}-\frac{28i}{149}$
15) $\frac{1}{3}-\frac{i}{3}$
16) $\frac{64}{61}-\frac{28i}{61}$
17) $-\frac{1}{4}-\frac{i}{4}$
18) $-\frac{10}{13}-\frac{2i}{13}$
19) $-\frac{11}{26}+\frac{29i}{26}$
20) $\frac{61}{149}-\frac{2i}{149}$

Conjugate

Change the sign of the imaginary part – for example the conjugate of -4+3i is -4-3i

Solving simple equations using Complex Numbers

1. Find the values of x and y in each of the following:
   (i) $(x+2)+i(y-1)=6-2i$
   (ii) $5-i=x+(5-2y)i$
   (iii) $(x+iy)+3(2-3i)=6-10i$
   (iv) $3(2x+3yi)+4-6i=2-5i$
   (v) $(2x-4)+5i=(x-3)+yi$
   (vi) $2(x+yi)+3(4-6i)=7-3i$
   (vii) $(x+2)+(y-4)i=(2y+1)+(x-5)i$
   (viii) $(x+iy)+(y-ix)=10-2i$
2. (i) Find x and y if $3(x+yi)-4(xi-y)=6-3i$.
   (ii) Find a and b if $(a+bi)(5+i)=3-2i$
3. Evaluate x and y if $2x+5yi=(6+2i)(3-4i)$.
4. Find the value of a and b if
   $$(4a-2)+(a-4)i=(4-2b)+2bi.$$
5. Find the numbers x and y such that
   $$(2+3i)(x+yi)=1-i.$$
6. Solve for x and y in each of the following equations:
   (i) $(2x-1)+(x+y)i=(y-6)+(2y-4)i$
   (ii) $2(x+yi)=4(2+3i)-2(1-2i)$
   (iii) $(3-4i)(x+yi)=1-3i$.

Solve Quadratic Equations with Complex Roots

When a quadratic equation cannot be solved by factorisation the following formula can be used

> The equation $ax^2 + bx + c = 0$ has the roots given by
>
> $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note: The whole of the top of the right hand side, including –b, is divided by 2a. It is also called the quadratic or –b formula. If $b^2 - 4ac < 0$, then the number under the square root sign will be negative, and so the solutions will be complex numbers.

Example

Solve the equation $x^2 - 4x + 13$

$ax^2 + bx + c = 0$ => $a = 1$, $b = -4$, $c = 13$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)} = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2}$$

$x = 2 \pm 3i$

Therefore, the roots are $2 + 3i$ and $2 - 3i$

Note: Notice the roots occur in conjugate pairs. If one root of a quadratic equation is a complex number then the other root must also be complex and the conjugate of the first: i.e., if $3 - 4i$ is a root, then $3 + 4i$ is also a root,

if $-2 - 5i$ is a root, then $-2 + 5i$ is also a root

if $a + bi$ is a root, then $a - bi$ is also a root

Questions

Solve for each of the following equations:

a) $x^2 - 6x + 13 = 0$

b) $z^2 - 2z + 10 = 0$

c) $x^2 + 16 = 0$

d) $x^2 + 41 = 10x$

e) $5 = 2x - x^2$

Complex Numbers Worksheet

1. **For the complex number $-10+4i$, identify the real number and the imaginary number**.
2. **Write the conjugate of each. Then plot all eight complex numbers in the same complex plane.**
   A) $-2+4i$ B) $7i$ C) 5 D) $3-2i$
3. **Evaluate.** a) i^6 b) i^{11} c) i^{24}
4. **Write the expression as a complex number in standard form.**
   a) $(5+2i)+(3-2i)$ b) $-i+(7-5i)-3(2-3i)$ c) $(-2+4i)+(3-9i)$
   d) $(-2+4i)-(3+9i)$ e) $(5-2i)-2(3+i)$ f) $3i(6-5i)$
   g) $i(2+i)$ h) $(2+3i)(1-4i)$ i) $(-3+7i)(1-2i)$
   j) $(3-2i)^2$ k) $(2i)(1-4i)(1+i)$
5. **Write the expression as a complex number in standard form.**
   a) $\frac{5}{1+i}$ b) $\frac{3-3i}{4i}$ c) $\frac{-2-4i}{7i}$ d) $\frac{8+7i}{3-4i}$
   e) $\frac{4+4i}{2-9i}$
6. **Find the absolute value of the complex number.**
   a) $-2+5i$ b) $4-5i$ c) $1-5i$ d) $-2+i$
   e) $-5i$
7. **Solve each equation.**
   a) $4x^2+20=0$ b) $4x^2+5=-7$ c) $x^2+4x=-20$ d) $x^2=8x-35$
   e) $x^2+4x=-29$ f) $3(x+4)^2=-27$ g) $8r^2+4r+5=0$
   h) $6p^2-8p=-3$

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3. Evaluate. a) i^6 b) i^{11} c) i^{24}

4. Write the expression as a complex number in standard form.

a) $(5+2i)+(3-2i)$ b) $-i+(7-5i)-3(2-3i)$ c) $(-2+4i)+(3-9i)$

d) $(-2+4i)-(3+9i)$ e) $(5-2i)-2(3+i)$ f) $3i(6-5i)$

g) $i(2+i)$ h) $(2+3i)(1-4i)$ i) $(-3+7i)(1-2i)$

j) $(3-2i)^2$ k) $(2i)(1-4i)(1+i)$

5. Write the expression as a complex number in standard form.

a) $\frac{5}{1+i}$ b) $\frac{3-3i}{4i}$ c) $\frac{-2-4i}{7i}$ d) $\frac{8+7i}{3-4i}$

e) $\frac{4+4i}{2-9i}$

6. Find the absolute value of the complex number.

a) $-2+5i$ b) $4-5i$ c) $1-5i$ d) $-2+i$

e) $-5i$

7. Solve each equation.

a) $4x^2+20=0$ b) $4x^2+5=-7$ c) $x^2+4x=-20$ d) $x^2=8x-35$

e) $x^2+4x=-29$ f) $3(x+4)^2=-27$ g) $8r^2+4r+5=0$

h) $6p^2-8p=-3$

Answers

1. Real number: -10; Imaginary number: $4i$
2. A) $-2-4i$ B) $-7i$ C) 5 D) $3+2i$
3. a) -1 b) $-i$ c) 1
4. a) 8 b) $1+3i$ c) $1-5i$ d) $-5-5i$ e) $-1-4i$ f) $15+18i$ g) $-1+2i$ h) $14-5i$ i) $11+13i$ j) $13-12i$ k) $(2i)(1-3i-4i^2)=(2i)(5-3i)=(10i-6i^2)=6+10i$
5. a) $\frac{5-5i}{2}$ b) $\frac{3}{4}+\frac{3i}{4}$ c) $\frac{4}{7}-\frac{2i}{7}$ d) $\frac{52-11i}{25}$ e) $\frac{28+44i}{85}$
6. a) $\sqrt{29}$ b) $\sqrt{41}$ c) $\sqrt{26}$ d) $\sqrt{5}$ e) 5

7. a) $\pm i\sqrt{5}$ b) $\pm i\sqrt{3}$ c) $-2\pm 4i$ d) $4\pm i\sqrt{19}$ e) $-2\pm 5i$ f) $-4\pm 3i$
g) $-\frac{1}{4}\pm\frac{3i}{4}$ h) $\frac{2}{3}\pm\frac{i\sqrt{2}}{6}$

Answers

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Answers

1. Real number: -10; Imaginary number: $4i$
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Various Exercises

1. If $z_1 = 2 - 3i$ and $z_2 = -2 + i$, express in the form $a + bi$:

(i) $z_1 + z_2$ (ii) $2z_1 - z_2$ (iii) $z_1.z_2$ (iv) iz_1 (v) $\frac{z_1}{z_2}$.

2. If $z_1 = 3 - 4i$ and $z_2 = 2 + 3i$, find

(i) $|z_1|$ (ii) $|z_2|$ (iii) $|z_1 + z_2|$ (iv) $|z_1.z_2|$.

Verify that $|z_1|.|z_2| = |z_1.z_2|$

Investigate if $|z_1| + |z_2| = |z_1 + z_2|$

3. (a) If $z = 2 - 3i$, plot on an Argand diagram

(i) $2z$ (ii) $-3z$ (iii) iz (iv) z^2.

(b) Write $\frac{-1+3i}{4-3i}$ in the form $x + iy$, $y \in R$.

(c) Solve for x and y the equation:

$(x + iy) + (3 - i) = 2(1 - 3i) - (y - ix)$.

4. (a) Represent the complex numbers $z = 2 - 3i$ and $w = 1 + 2i$ on an Argand diagram.

What is (i) $|z|$ (ii) $|z + w|$?

(b) Express (i) zw (ii) $\frac{1}{z}$ in the form $a + ib$, a, $b \in R$.

(c) If $(3 - 5i) - (x + yi) = (6 + i) + (y - xi)$, x, $y \in R$, find x and y.

5. (a) (i) Verify that $z_1 = 2 + 3i$ is a root of the equation
$z^2 - 4z + 13 = 0$

(ii) Find the value of $|z_1|$

(iii) On an Argand diagram illustrate z_1 and z_2, the image of z_1 under central symmetry in the origin.

(b) If $a + bi = \frac{5}{1+2i}$, where a, $b \in R$, calculate the value of a and the value of b.

6. Verify that the complex number $z_1 = 3 - 2i$ is a root of the equation
$$z^2 - 6z + 13 = 0$$
and find z_2, the other root of the equation.

On an Argand diagram plot the complex numbers z_1 and z_2.
z_3 is the image of z_1 under central symmetry in z_2.
Express z_3 in the form $a + bi$ and plot it on an Argand diagram.

Investigate if $|z_1 - z_2| = |z_1| - |z_2|$.

Q7 Express in the form a+bi (below)

7. (a) If $z_1 = 3 + 2i$ and $z_2 = 3 - i$, express in the form

(i) z_1^2 (ii) $z_1 - z_2$ (iii) $\frac{z_1}{z_2}$

Find (iv) $|z_1|$ (v) $|z_2|$ (vi) $\left|\frac{z_1}{z_2}\right|$

Verify that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(b) If $2 - i$ is a root of the equation $z^2 + tz + k = 0$, $t, k \in R$, find the value of t and the value of k.

8. Verify that $5 - 3i$ is a root of the equation $z^2 - 10z + 34 = 0$ and write down the other root.

Let $z_1 = 5 - 2i$ and $z_2 = 4 + i$.
Find z_3 such that $z_1 - z_3 = z_2$.

Plot the three complex numbers z_1, iz_1 and $8 - z_3$ on an Argand diagram and verify that

$$|z_1 - 8 + z_3| = \sqrt{29}.$$

A circle is drawn on the Argand diagram having $8 - z_3$ as centre and $\sqrt{29}$ as radius.
Show that this circle contains (passes through) iz_1.

9. (a) Solve the equation $z^2 - 4z + 13 = 0$, $z \in C$, and plot the roots on an Argand diagram.

(b) If $z_1 = 3 - 4i$ and $z_2 = 3 + 4i$, show that

(i) $|z_1 . z_2| = |z_1|^2$ (ii) $|z_1 + z_2| < 2|z_1|$

(c) Find x and $y \in R$ such that $(4 + 3i)(x + yi) = 1 + 7i$.

10. (a) If $-3 + 2i$ is a root of the equation $z^2 + az + b = 0$, where $a, b \in R$, find the value of a and the value of b.

(b) Let $z_1 = 1 + 3i$ and $z_2 = 1 + i$.
Let $z_3 = z_2 + iz_1$ and $z_4 = \frac{z_1}{z_2}$

Express z_3 and z_4 in the form $x + yi$.

Plot z_3 and z_4 on an Argand diagram and investigate if the image of z_3 under central symmetry in the origin is the same as the image of z_4 under axial symmetry in the real axis (i.e. x-axis).

11. Let $z = 3 - 2i$, where $i = \sqrt{-1}$.

Express in the form $a + bi$, z^2 and $\frac{13}{z}$.

Plot i, z and $\frac{13}{z}$ on an Argand Diagram.

$z = 3 - 2i$ is a solution of the equation

$$z^2 + bz + c = 0, \quad b, c \in \mathbf{R}.$$

Find the value of b and the value of c.

Investigate if $|z| = |iz|$.

Give a geometrical interpretation for your answer.

12. (a) On the Argand diagram plot the point p which represents the complex number $1 + 2i$. What complex number in each of the following is represented by the image of p under

(i) the central symmetry in the origin

(ii) the axial symmetry in the real axis (i.e. the x-axis) *after* the central symmetry in the origin

(iii) the axial symmetry in the imaginary axis (i.e. the y-axis)?

(b) Find the values of $x, y \in \mathbf{R}$ such that

$(x - iy) + (y + ix) = 1 - 5i$ where $i = \sqrt{-1}$.

Using these values of x and y express $\frac{x - iy}{y + ix}$

in the form $a + ib$, where $a, b \in \mathbf{R}$, and hence, or otherwise, find the value of

$$\left|\frac{x - iy}{y + ix}\right|.$$

13. Represent the complex numbers $z_1 = 2 + i$ and $z_2 = 2 - i$ on an Argand diagram.
Calculate $\frac{1}{2}(z_1 + z_2)$.
Verify that $|z_1.z_2| = |z_1|.|z_2|$
Express $\frac{z_1}{z_2}$ in the form $a + ib$ and find k such that $|z_1| = k\left|\frac{z_1}{z_2}\right|$.

14. (a) Find the real part and the imaginary part of

$$\frac{2 + 4i}{3 - 2i}.$$

(b) Find the value of x and the value of y if

$$x(3 + 4i) + 5 = y(1 + 2i).$$

(c) If $z_1 = 3 + 3i$ and $z_2 = 2 - 2i$, find the image of z_1 under central symmetry in z_2.

If $z_2 - tz_1 = ki$, where $t, k \in R$, find t and k.

Modulus

Let z = a + ib be a complex number. Then, the modulus of a complex number z, denoted by |z|, is defined to be the non-negative real number. $\sqrt{a^2+b^2}$

modulus of a complex number z = |z| = $\sqrt{Re(z)^2+Im(z)^2}$

where Real part of complex number = Re(z) = a and

Imaginary part of complex number =Im(z) =b

$$|z| = \sqrt{a^2+b^2}.$$

Example :

(i) z = 5 + 6i so |z| = $\sqrt{5^2+6^2}$ = $\sqrt{25+36}$= $\sqrt{61}$

(ii) z = 8 + 5i so |z| = $\sqrt{8^2+5^2}$ = $\sqrt{64+25}$= $\sqrt{89}$

(iii) z = 3 - i so |z| = $\sqrt{3^2+(-1)^2}$ = $\sqrt{9+1}$= $\sqrt{10}$

(iv) z = 1 + $\sqrt{5}$i so |z| = $\sqrt{1^2+\sqrt{5}^2}$ = $\sqrt{64+25}$= $\sqrt{89}$

(v) -6 + 2i so |z| = $\sqrt{(-6)^2+2^2}$ = $\sqrt{36+4}$= $\sqrt{40}$

(vi) -8 + 6i so |z| = $\sqrt{(-8)^2+6^2}$ = $\sqrt{64+36}$= $\sqrt{100}$ = 10

(vii) 12 - 5i so |z| = $\sqrt{12^2+(-5)^2}$ = $\sqrt{144+25}$= $\sqrt{169}$ =13

Modulus and Argument of a Complex Number

In this unit you are going to learn about the **modulus** and **argument** of a complex number. These are quantities which can be recognised by looking at an Argand diagram. Recall that any complex number, z, can be represented by a point in the complex plane as shown in Figure 1.

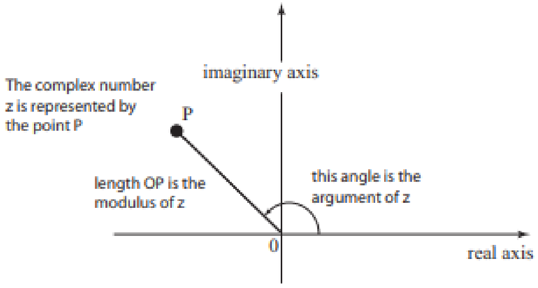


Figure 1. The complex number z is represented by point P. Its modulus and argument are shown.

We can join point P to the origin with a line segment, as shown. We associate with this line segment two important quantities. The length of the line segment, that is OP, is called the **modulus** of the complex number. The angle from the positive axis to the line segment is called the **argument** of the complex number, z.

The modulus and argument are fairly simple to calculate using trigonometry.

Example. Find the modulus and argument of $z = 4 + 3\text{i}$.

Solution. The complex number $z = 4 + 3\text{i}$ is shown in Figure 2. It has been represented by the point Q which has coordinates $(4, 3)$. The modulus of z is the length of the line OQ which we can find using Pythagoras' theorem.

$$(OQ)^2 = 4^2 + 3^2 = 16 + 9 = 25$$

and hence $OQ = 5$.

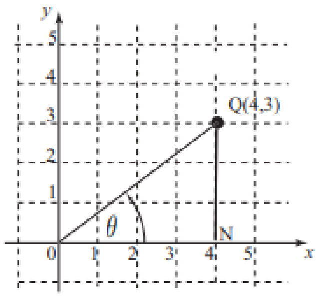


Figure 2. The complex number $z = 4 + 3\text{i}$.

Hence the modulus of $z = 4 + 3\text{i}$ is 5. To find the argument we must calculate the angle between the x axis and the line segment OQ. We have labelled this θ in Figure 2.

By referring to the right-angled triangle OQN in Figure 2 we see that

$$\tan\theta = \frac{3}{4}$$

$$\theta = \tan^{-1}\frac{3}{4} = 36.97^\circ$$

To summarise, the modulus of $z = 4 + 3\text{i}$ is 5 and its argument is $\theta = 36.97^\circ$. There is a special symbol for the modulus of z; this is $|z|$. So, in this example, $|z| = 5$. We also have an abbreviation for argument: we write $\arg(z) = 36.97^\circ$.

When the complex number lies in the first quadrant, calculation of the modulus and argument is straightforward. For complex numbers outside the first quadrant we need to be a little bit more careful. Consider the following example.

Example.

Find the modulus and argument of $z = 3 - 2\text{i}$.

Solution. The Argand diagram is shown in Figure 3. The point P with coordinates $(3, -2)$ represents $z = 3 - 2\text{i}$.

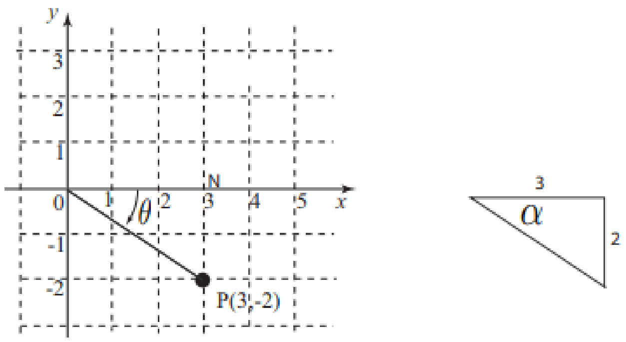


Figure 3. The complex number $z = 3 - 2\text{i}$.

We use Pythagoras' theorem in triangle ONP to find the modulus of z:

$$(OP)^2 = 3^2 + 2^2 = 13$$

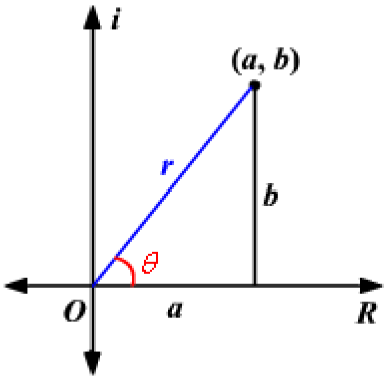
$$OP = \sqrt{13}$$

Using the symbol for modulus, we see that in this example $|z| = \sqrt{13}$.

We must be more careful with the argument. When the angle θ shown in Figure 3 is measured in a clockwise sense convention dictates that the angle is negative. We can find the size of the angle by referring to the right-angled triangle shown. In that triangle $\tan\alpha = \frac{2}{3}$ so that $\alpha = \tan^{-1}\frac{2}{3} = 33.67^\circ$. This is not the argument of z. The argument of z is $\theta = -33.67^\circ$. We often write this as $\arg(z) = -33.67^\circ$.

Complex Numbers in Polar Form

$$r(cos\theta, isin\theta)$$



Example:

Express the complex number in polar form.

$5 + 2i$

The polar form of a complex number $z = a + bi$ is $z = r(\cos\theta + i\sin\theta)$.

So, first find the absolute value of r .

$$\begin{aligned} r = |z| &= \sqrt{a^2 + b^2} \\ &= \sqrt{5^2 + 2^2} \\ &= \sqrt{25 + 4} \\ &= \sqrt{29} \\ &\approx 5.39 \end{aligned}$$

Now find the argument θ .

Since $a > 0$, use the formula $\theta = \tan^{-1}\left(\frac{b}{a}\right)$.

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{2}{5}\right) \\ &\approx 0.38 \end{aligned}$$

Note that here θ is measured in radians.

Therefore, the polar form of $5 + 2i$ is about $5.39(\cos(0.38) + i\sin(0.38))$.

Complex Numbers in General Polar Form

$r(\cos(\theta + 2n\pi), i\sin(\theta + 2n\pi))$

Exercises

Write each of the following in Polar and General Polar

Express each of these complex numbers in the form $r(\cos\theta + i\sin\theta)$, where $i^2 = -1$:

(i) $-1+i$ **(ii)** $-\sqrt{3}-i$ **(iii)** $\frac{1}{2}+\frac{\sqrt{3}}{2}i$ **(iv)** $-6i$.

Solutions on next page

Solution:

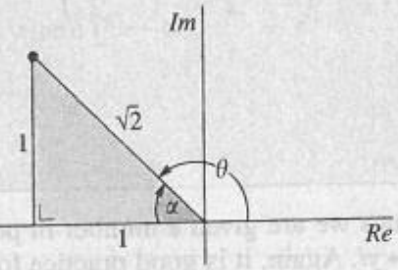
(i) $-1+i=(-1,\ 1)$

$r=|-1+i|=\sqrt{(-1)^2+(1)^2}=\sqrt{1+1}=\sqrt{2}$

$\tan\alpha=\dfrac{1}{1}=1 \quad\Rightarrow\quad \alpha=\dfrac{\pi}{4}$

$\therefore\quad \theta=\pi-\dfrac{\pi}{4}=\dfrac{3\pi}{4}$

$\therefore\quad -1+i=\sqrt{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)$



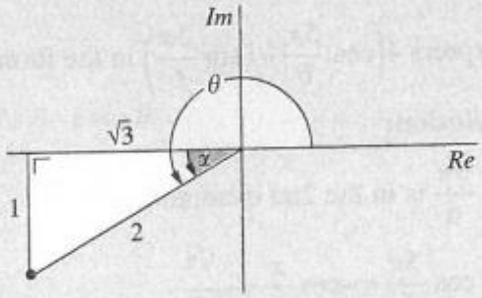
(ii) $-\sqrt{3}-i=(-\sqrt{3},-1)$

$r=|-\sqrt{3}-i|=\sqrt{(-\sqrt{3})^2+(-1)^2}=\sqrt{3+1}=\sqrt{4}=2$

$\tan\alpha=\dfrac{1}{\sqrt{3}} \quad\Rightarrow\quad \alpha=\dfrac{\pi}{6}$

$\therefore\quad \theta=\pi+\dfrac{\pi}{6}=\dfrac{7\pi}{6}$

$\therefore\quad -\sqrt{3}-i=2\left(\cos\dfrac{7\pi}{6}+i\sin\dfrac{7\pi}{6}\right)$

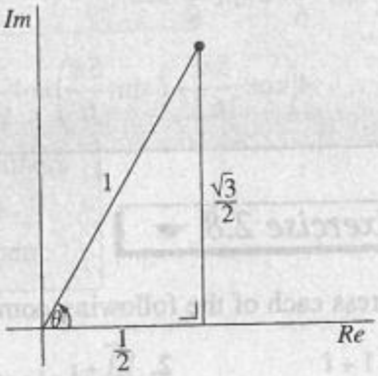


(iii) $\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i=\left(\dfrac{1}{2},\ \dfrac{\sqrt{3}}{2}\right)$

$r=\left|\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}\right|=\sqrt{\left(\dfrac{1}{2}\right)^2+\left(\dfrac{\sqrt{3}}{2}\right)^2}=\sqrt{\dfrac{1}{4}+\dfrac{3}{4}}=\sqrt{1}=1$

$\tan\theta=\dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}=\sqrt{3} \quad\Rightarrow\quad \theta=\dfrac{\pi}{3}$

$\therefore\quad \dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i=\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}$

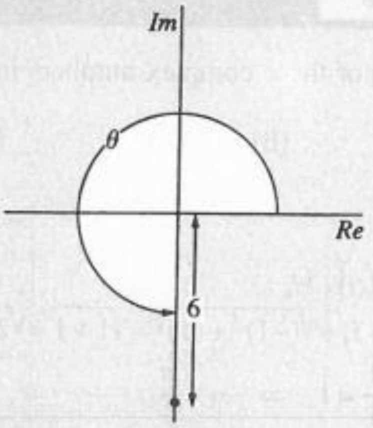


(iv) $-6i=0-6i=(0,-6)$

$r=|0-6i|=\sqrt{0^2+(-6)^2}=\sqrt{0+36}=\sqrt{36}=6$

$\theta=\dfrac{3\pi}{2}$

$\therefore\quad -6i=6\left(\cos\dfrac{3\pi}{2}+i\sin\dfrac{3\pi}{2}\right)$



Exercise

Express $4\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$ in the form $x+yi$.

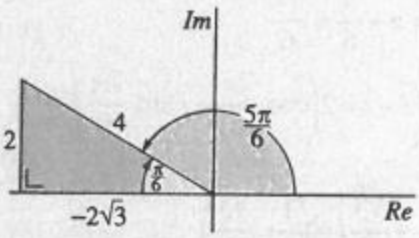
Express $4\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$ in the form $x+yi$.

Solution:

$\frac{5\pi}{6}$ is in the 2nd quadrant, so:

$\cos\frac{5\pi}{6}=-\cos\frac{\pi}{6}=-\frac{\sqrt{3}}{2}$

$\sin\frac{5\pi}{6}=\sin\frac{\pi}{6}=\frac{1}{2}$



$\therefore\quad 4\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)=4\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=-2\sqrt{3}+2i$

More Exercises

Express each of the following complex numbers in the form $r(\cos\theta+i\sin\theta)$, where $i^2=-1$:

1. $1+i$ **2.** $\sqrt{3}+i$ **3.** -5 **4.** $3i$

5. $-2i$ **6.** $-1-\sqrt{3}i$ **7.** $1-i$ **8.** $2-2i$

9. $-\sqrt{2}-\sqrt{2}i$ **10.** $-3+\sqrt{3}i$ **11.** $\frac{1}{2}-\frac{\sqrt{3}}{2}i$ **12.** $-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$

Express each of the following in the form $a+bi$:

13. $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$ **14.** $\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ **15.** $6\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

16. $2\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$ **17.** $10\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$ **18.** $2\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$