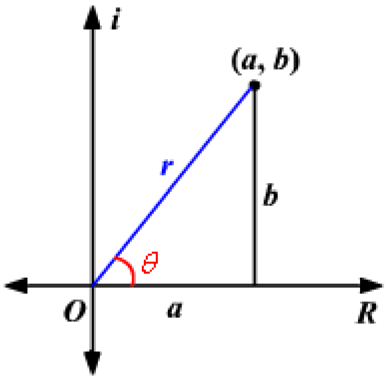
Complex Numbers in Polar Form

$$r(cos\theta, isin\theta)$$



Example:

Express the complex number in polar form.

$5 + 2i$

The polar form of a complex number $z = a + bi$ is $z = r(\cos\theta + i\sin\theta)$.

So, first find the absolute value of r .

$$\begin{aligned} r = |z| &= \sqrt{a^2 + b^2} \\ &= \sqrt{5^2 + 2^2} \\ &= \sqrt{25 + 4} \\ &= \sqrt{29} \\ &\approx 5.39 \end{aligned}$$

Now find the argument θ .

Since $a > 0$, use the formula $\theta = \tan^{-1}\left(\frac{b}{a}\right)$.

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{2}{5}\right) \\ &\approx 0.38 \end{aligned}$$

Note that here θ is measured in radians.

Therefore, the polar form of $5 + 2i$ is about $5.39(\cos(0.38) + i\sin(0.38))$.

Complex Numbers in General Polar Form

$r(\cos(\theta + 2n\pi), i\sin(\theta + 2n\pi))$

Exercises

Write each of the following in Polar and General Polar

Express each of these complex numbers in the form $r(\cos\theta + i\sin\theta)$, where $i^2 = -1$:

(i) $-1+i$ (ii) $-\sqrt{3}-i$ (iii) $\frac{1}{2}+\frac{\sqrt{3}}{2}i$ (iv) $-6i$.

Solutions on next page

Solution:

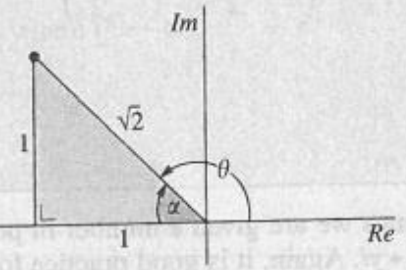
(i) $-1+i=(-1,\ 1)$

$r=|-1+i|=\sqrt{(-1)^2+(1)^2}=\sqrt{1+1}=\sqrt{2}$

$\tan\alpha=\frac{1}{1}=1 \quad\Rightarrow\quad \alpha=\frac{\pi}{4}$

$\therefore\quad \theta=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$

$\therefore\quad -1+i=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$



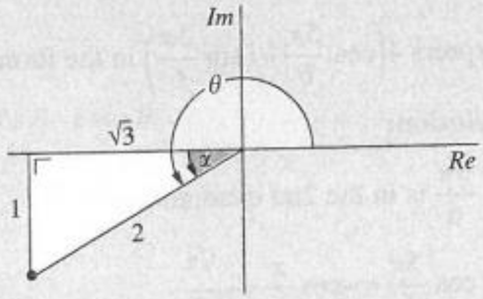
(ii) $-\sqrt{3}-i=(-\sqrt{3},-1)$

$r=|-\sqrt{3}-i|=\sqrt{(-\sqrt{3})^2+(-1)^2}=\sqrt{3+1}=\sqrt{4}=2$

$\tan\alpha=\frac{1}{\sqrt{3}} \quad\Rightarrow\quad \alpha=\frac{\pi}{6}$

$\therefore\quad \theta=\pi+\frac{\pi}{6}=\frac{7\pi}{6}$

$\therefore\quad -\sqrt{3}-i=2\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)$

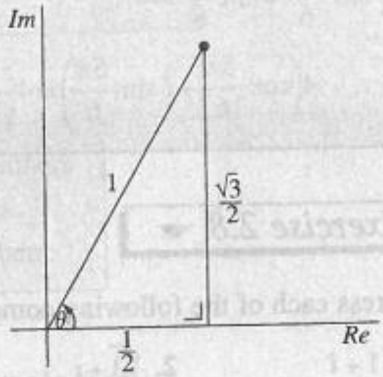


(iii) $\frac{1}{2}+\frac{\sqrt{3}}{2}i=\left(\frac{1}{2},\ \frac{\sqrt{3}}{2}\right)$

$r=\left|\frac{1}{2}+\frac{\sqrt{3}}{2}\right|=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=\sqrt{1}=1$

$\tan\theta=\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}=\sqrt{3} \quad\Rightarrow\quad \theta=\frac{\pi}{3}$

$\therefore\quad \frac{1}{2}+\frac{\sqrt{3}}{2}i=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$

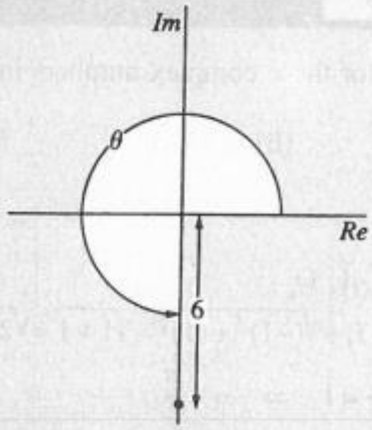


(iv) $-6i=0-6i=(0,-6)$

$r=|0-6i|=\sqrt{0^2+(-6)^2}=\sqrt{0+36}=\sqrt{36}=6$

$\theta=\frac{3\pi}{2}$

$\therefore\quad -6i=6\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$



Exercise

Express $4\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$ in the form $x+yi$.

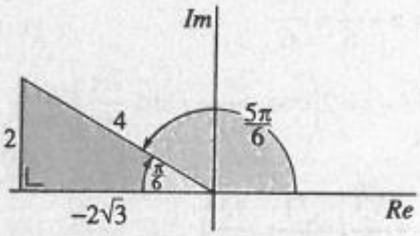
Express $4\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$ in the form $x+yi$.

Solution:

$\frac{5\pi}{6}$ is in the 2nd quadrant, so:

$\cos\frac{5\pi}{6}=-\cos\frac{\pi}{6}=-\frac{\sqrt{3}}{2}$

$\sin\frac{5\pi}{6}=\sin\frac{\pi}{6}=\frac{1}{2}$



$\therefore\quad 4\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)=4\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=-2\sqrt{3}+2i$

More Exercises

Express each of the following complex numbers in the form $r(\cos\theta+i\sin\theta)$, where $i^2=-1$:

1. $1+i$ **2.** $\sqrt{3}+i$ **3.** -5 **4.** $3i$

5. $-2i$ **6.** $-1-\sqrt{3}i$ **7.** $1-i$ **8.** $2-2i$

9. $-\sqrt{2}-\sqrt{2}i$ **10.** $-3+\sqrt{3}i$ **11.** $\frac{1}{2}-\frac{\sqrt{3}}{2}i$ **12.** $-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$

Express each of the following in the form $a+bi$:

13. $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$ **14.** $\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ **15.** $6\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

16. $2\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$ **17.** $10\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$ **18.** $2\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$