4-6 Integration

Integration is used to find areas under curves. Integration is the reversal of differentiation hence functions can be integrated by indentifying the anti-derivative.

However, we will learn the process of integration as a set of rules rather than identifying anti-derivatives.

Terminology

- Indefinite and Definite integrals

There are two types of integrals: Indefinite and Definite.

Indefinite integrals are those with no limits and definite integrals have limits.

When dealing with indefinite integrals you need to add a constant of integration. For example, if integrating the function f(x) with respect to x:

$\int (f\ x)\ dx = g(x) + C$, where g(x) is the integrated function.

- C is an arbitrary constant called the constant of integration.
- dx indicates the variable with respect to which we are integrating, in this case, x.
- The function being integrated, f(x), is called the integrand.

Rules on next page – see Log Tables/Mathematics Tables

- The Power Rule

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \text{ provided that } n \neq -1$$

Examples: $\int x^5 \, dx = \frac{x^6}{6} + C$

$$\int x^{-4} \, dx = \frac{x^{-3}}{-3} + C$$

- When n = -1

$$\int x^{-1} \, dx = \int \frac{1}{x} dx = \ln x + C$$

- Constant rule

$$\int k \, dx = kx + C \qquad \text{where k is a constant}$$

Example: $\int 2 \, dx = 2x + C$

- Exponentials

$$\int e^{kx} \, dx = \frac{1}{k} e^{kx} + C$$

Example: $\int e^{9x} \, dx = \frac{1}{9} e^{9x} + C$

$$\int e^x \, dx = e^x + C$$

- Trig functions
  - Cos

$$\int \cos(x) \, dx = \sin(x) + C$$

$$\int \cos(kx) \, dx = \frac{1}{k} \sin(kx) + C \qquad \text{where k is a constant}$$

Example: $\int \cos(12x) \, dx = \frac{1}{12} \sin(12x) + C$

- Sin

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + C \qquad \text{where k is a constant}$$

Example: $\int \sin(10x)\,dx = -\frac{1}{10}\cos(10x) + C$

$$\int \sin(-5x)\,dx = \frac{1}{5}\cos(-5x) + C$$

Linearity

Suppose *f(x)* and *g(x)* are two functions in terms of x, then:

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

Additionally, if A and B are constants, then

$$\int [Af(x) \pm Bg(x)]dx = A\int f(x)dx \pm B\int g(x)dx$$

Examples:

$$\int (2x^4 + 3x^5)dx = \int 2x^4\,dx + \int 3x^5\,dx$$

$$= 2\int x^4\,dx + 3\int x^5\,dx$$

$$= 2\left(\frac{x^5}{5}\right) + 3\left(\frac{x^6}{6}\right) + C$$

$$= \frac{2x^5}{5} + \frac{x^6}{2} + C$$

$$\int (5\cos(3x) - 3e^{7x})dx = \int 5\cos(3x)dx - \int 3e^{7x}\,dx$$

$$= 5\int \cos(3x)dx - 3\int e^{7x}\,dx$$

$$= 5\left(\frac{1}{3}\sin(3x)\right) - 3\left(\frac{1}{7}e^{7x}\right)$$

$$= \frac{5}{3}\sin(3x) - \frac{3}{7}e^{7x}$$

Questions (General rules)**:**

Integrate the following functions:

1. $\int \left(x^6 - x^{\frac{3}{2}} + \frac{1}{x^5}\right)dx$
2. $\int \left(3x^8 + x - 5\right)dx$
3. $\int \left(9x^2 - 3x^{-1}\right)dx$
4. $\int \left(\sin(4x) + e^{3x}\right)dx$
5. $\int \left(\cos(7x) + 7x^2\right)dx$

(Solutions on page 8)

Definite Integrals

Earlier we saw that

$$\int f(x)dx = g(x) + C$$

Suppose now we are given limits, i.e.

$$\int_a^b f(x)dx = g(x) + C$$

(where a is the lower limit and b is the upper limit)

This can be interpreted as:

(value of g(x) + C at x = b) – (value of g(x) + C at x = a)

In other words, since C will cancel out:

$$\int_a^b f(x)dx = g(b) - g(a)$$

Examples

1. $\int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}\left[x^3\right]_0^1 = \frac{1}{3}\{(1)^3 - (0)^3\} = \frac{1}{3}(1 - 0) = \frac{1}{3}$

2. $\int_1^3 (2x+1)dx = \left[\frac{2x^2}{2} + x\right]_1^3 = \left[x^2 + x\right]_1^3 = \{(3^2 + 3) - (1^2 + 1)\}$

   $= \{(9 + 3) - (1 + 1)\} = 12 - 2 = 10$

3. $\int_0^{\frac{\pi}{2}} \cos(x)dx = [\sin(x)]_0^{\frac{\pi}{2}} = \{(\sin(\frac{\pi}{2})) - (\sin(0))\}$

   $= 1 - 0 = 1$

Questions (Definite integrals)**:**

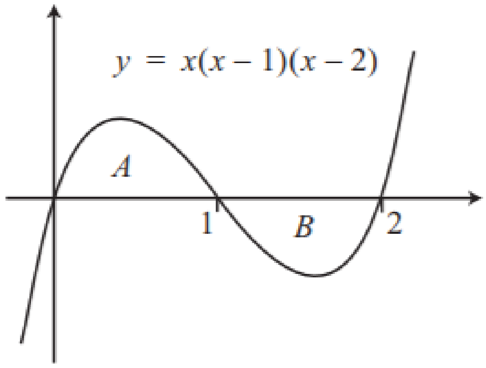
Integrate the following functions:

1. $\int_1^2 (3x^2 - 2x + 5)dx$
2. $\int_0^1 e^{7x}\,dx$
3. $\int_0^{\pi} \sin(2x)dx$
4. $\int_1^4 (12e^{4x} + 4\sqrt{x})dx$

2. The area between a curve and the x-axis

Let us begin by exploring the following question: 'Calculate the areas of the segments contained between the x-axis and the curve $y = x(x-1)(x-2)$.'

In order to answer this question, it seems reasonable for us to draw a sketch of the curve, as there is no mention of the ordinates, or values, of x. To make the sketch, we see first that the curve crosses the x-axis when $y = 0$, in other words when $x = 0$, $x = 1$, and $x = 2$. Next, when x is large and positive, we see that y is also large and positive. Finally, when x is large and negative, we see that y is also large and negative. So if we join up these features that we have found on the graph, we can see that the curve looks like this.



Now the areas required are obviously the area A between $x = 0$ and $x = 1$, and the area B between $x = 1$ and $x = 2$. But there is a marked difference between these two areas in terms of their position. The area A is above the x-axis, whereas the area B is below it. In previous units we have talked only about calculating areas using integration when the curve, and thus the area, is above the x-axis. Does the position of the curve make any difference to the area?

In this example, we shall play safe and calculate each area separately. We know that the area A is given by the integral from $x = 0$ to $x = 1$ of the curve $y = x(x-1)(x-2) = x^3 - 3x^2 + 2x$; thus

$$\begin{aligned}
A &= \int_0^1 y\,\mathrm{d}x \\
&= \int_0^1 (x^3 - 3x^2 + 2x)\mathrm{d}x \\
&= \left[\frac{x^4}{4} - \frac{3x^3}{3} + \frac{2x^2}{2}\right]_0^1 \\
&= \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 \\
&= [\tfrac{1}{4} - 1 + 1] - [\tfrac{0}{4} - 0 + 0] \\
&= \tfrac{1}{4}.
\end{aligned}$$

Area B should be given by a similar integral, except that now the limits of integration are from $x = 1$ to $x = 2$:

$$\begin{aligned} B &= \int_1^2 y\,\mathrm{d}x \\ &= \int_1^2 (x^3 - 3x^2 + 2x)\mathrm{d}x \\ &= \left[\frac{x^4}{4} - \frac{3x^3}{3} + \frac{2x^2}{2}\right]_1^2 \\ &= \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2 \\ &= [\tfrac{16}{4} - 8 + 4] - [\tfrac{1}{4} - 1 + 1] \\ &= 0 - \tfrac{1}{4} \\ &= -\tfrac{1}{4}. \end{aligned}$$

Now the two integrals have the same magnitude, but area A is above the x-axis and area B is below the x-axis; and, as we see, the sign of the value B is negative. The actual value of the area is $+\frac{1}{4}$, so why does our calculation give a negative answer?

In the unit on Integration as Summation, when we summed the small portions of area we were evaluating $\sum y\,\delta x$. Now δx was defined as a small positive increment in x, but y was simply the y-value at the ordinate x. Clearly this y-value will be negative if the curve is below the x-axis, so in this case the quantity $y\,\delta x$ will be *minus* the value of the area. So in our example, we were adding up a lot of values equal to *minus* the area, as the curve is wholly below the x-axis between $x = 1$ and $x = 2$. For this reason, the calculation gives a negative answer which is minus the value of the area.

Now suppose for a moment that the question had asked us instead to find the *total* area enclosed by the curve and the x-axis. We know the answer to this question. The total area is the area of A, $\frac{1}{4}$ of a unit of area, added to actual value of the area B, which is another $\frac{1}{4}$ of a unit of area. Thus the total area enclosed by the curve and the x-axis is $\frac{1}{2}$ of a unit of area. But suppose we had decided to work this out by finding the value of the integral between $x = 0$ and $x = 2$, without drawing a sketch. What answer would we get? We would find

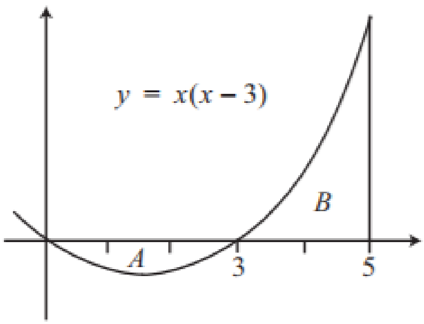
$$\begin{aligned} \text{'Area'} &= \int_0^2 y\,\mathrm{d}x \\ &= \int_0^2 (x^3 - 3x^2 + 2x)\mathrm{d}x \\ &= \left[\frac{x^4}{4} - \frac{3x^3}{3} + \frac{2x^2}{2}\right]_0^2 \\ &= \left[\frac{x^4}{4} - x^3 + x^2\right]_0^2 \\ &= [\tfrac{16}{4} - 8 + 4] - [\tfrac{0}{4} - 0 + 0] \\ &= 0 - 0 \end{aligned}$$

Example

Find the area between the curve $y = x(x-3)$ and the ordinates $x = 0$ and $x = 5$.

Solution

If we set $y = 0$ we see that $x(x-3) = 0$, and so $x = 0$ or $x = 3$. Thus the curve cuts the x-axis at $x = 0$ and at $x = 3$. The x^2 term is positive, and so we know that the curve forms a U-shape as shown below.



From the graph, we can see that we need to calculate the area A between the curve, the x-axis and the ordinates $x = 0$ and $x = 3$ first, and that we should expect this integral to give a negative answer because the area is wholly below the x-axis:

$$\begin{aligned} A &= \int_0^3 y\,\mathrm{d}x \\ &= \int_0^3 (x^2 - 3x)\mathrm{d}x \\ &= \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_0^3 \\ &= [\tfrac{27}{3} - \tfrac{3\times 9}{2}] - [\tfrac{0}{3} - \tfrac{3\times 0}{2}] \\ &= [9 - \tfrac{27}{2}] - [0] \\ &= -4\tfrac{1}{2}\,. \end{aligned}$$

Next, we need to calculate the area B between the curve, the x-axis, and the ordinates $x = 3$ and $x = 5$:

$$\begin{aligned} B &= \int_3^5 y\,\mathrm{d}x \\ &= \int_3^5 (x^2 - 3x)\mathrm{d}x \\ &= \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_3^5 \\ &= [\tfrac{125}{3} - \tfrac{3\times 25}{2}] - [\tfrac{27}{3} - \tfrac{3\times 9}{2}] \\ &= 41\tfrac{2}{3} - 37\tfrac{1}{2} - 9 + 13\tfrac{1}{2} \\ &= 8\tfrac{2}{3}\,. \end{aligned}$$

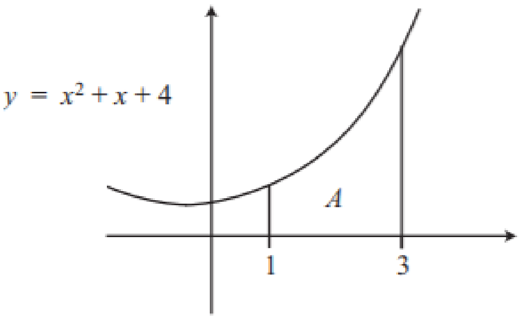
So the total actual area is $4\frac{1}{2} + 8\frac{2}{3} = 13\frac{1}{6}$ units of area.

Example

Find the area bounded by the curve $y = x^2 + x + 4$, the x-axis and the ordinates $x = 1$ and $x = 3$.

Solution

If we set $y = 0$ we obtain the quadratic equation $x^2 + x + 4 = 0$, and for this quadratic $b^2 - 4ac = 1 - 16 = -15$ so that there are no real roots. This means that the curve does not cross the x-axis. Furthermore, the coefficient of x^2 is positive and so the curve is U-shaped. When $x = 0$, $y = 4$ and so the curve looks like this.



The required area A is entirely above the x-axis and so we can simply evaluate the integral between the required limits:

$$
\begin{aligned}
A &= \int_1^3 y\,\mathrm{d}x \\
&= \int_1^3 (x^2 + x + 4)\mathrm{d}x \\
&= \left[\frac{x^3}{3} + \frac{x^2}{2} + 4x\right]_1^3 \\
&= [\tfrac{27}{3} + \tfrac{9}{2} + 12] - [\tfrac{1}{3} + \tfrac{1}{2} + 4] \\
&= 25\tfrac{1}{2} - 4\frac{5}{6}
\end{aligned}
$$

which equals $20\frac{2}{3}$ units of area.

Exercises

1. Find the area enclosed by the given curve, the x-axis, and the given ordinates.

   (a) The curve $y = x$, from $x = 1$ to $x = 3$.

   (b) The curve $y = x^2 + 3x$, from $x = 1$ to $x = 3$

   (c) The curve $y = x^2 - 4$ from $x = -2$ to $x = 2$

   (d) The curve $y = x - x^2$ from $x = 0$ to $x = 2$

2. Find the area contained by the curve $y = x(x - 1)(x + 1)$ and the x-axis.

3. Calculate the value of
$$\int_{-1}^{1} x(x - 1)(x + 1)\mathrm{d}x\,.$$
Compare your answer with that obtained in question 3, and explain what has happened.

4. Calculate the value of
$$\int_0^6 (4x - x^2)\mathrm{d}x\,.$$
Explain your answer.

The area between two curves

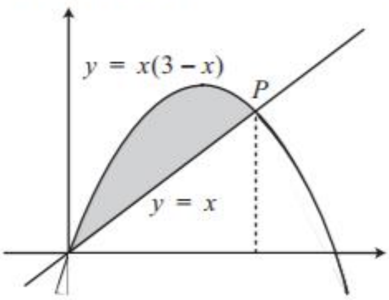
A similar technique to the one we have just used can also be employed to find the areas sandwiched between curves.

Example

Calculate the area of the segment cut from the curve $y = x(3-x)$ by the line $y = x$.

Solution

Sketching both curves on the same axes, we can see by setting $y = 0$ that the curve $y = x(3-x)$ cuts the x-axis at $x = 0$ and $x = 3$. Furthermore, the coefficient of x^2 is negative and so we have an inverted U-shape curve. The line $y = x$ goes through the origin and meets the curve $y = x(3-x)$ at the point P. It is this point that we need to find first of all.



At P the y co-ordinates of both curves are equal. Hence:

$$\begin{aligned} x(3-x) &= x \\ 3x - x^2 &= x \\ 2x - x^2 &= 0 \\ x(2-x) &= 0 \end{aligned}$$

so that either $x = 0$, the origin, or else $x = 2$, the x co-ordinate of the point P.

We now need to find the shaded area in the diagram. To do this we need the area under the upper curve, the graph of $y = x(3-x)$, between the x-axis and the ordinates $x = 0$ and $x = 2$. Then we need to subtract from this the area under the lower curve, the line $y = x$, and between the x-axis and the ordinates $x = 0$ and $x = 2$.

The area under the curve is

$$\begin{aligned} \int_0^2 y\,\mathrm{d}x &= \int_0^2 x(3-x)\mathrm{d}x \\ &= \int_0^2 (3x - x^2)\mathrm{d}x \\ &= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^2 \\ &= [6 - \tfrac{8}{3}] - [0] \\ &= 3\tfrac{1}{3}, \end{aligned}$$

and the area under the straight line is

$$\begin{aligned} \int_0^2 y\,\mathrm{d}x &= \int_0^2 x\,\mathrm{d}x \\ &= \left[\frac{x^2}{2}\right]_0^2 \\ &= [2] - [0] \\ &= 2. \end{aligned}$$

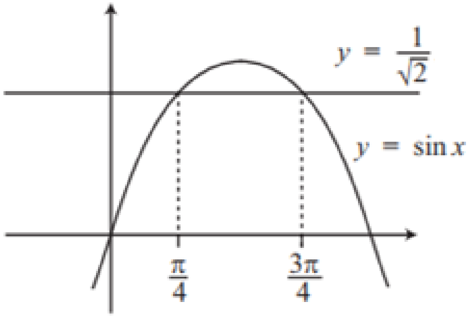
Thus the shaded area is $3\frac{1}{3} - 2 = 1\frac{1}{3}$ units of area.

Example

Find the area of the segment cut off from the curve $y = \sin x$, $0 \le x \le \pi$ by the line $y = 1/\sqrt{2}$.

Solution

We begin by sketching the curve $y = \sin x$ over the given range, and then the line $y = 1/\sqrt{2}$ on the same axes and for the same range.



We can see that we need to find the two points P and Q where the line intersects the curve. This will be when

$$\sin x = \frac{1}{\sqrt{2}},$$

in other words when $x = \pi/4$ and $x = 3\pi/4$. So we need to calculate the area under the curve between the x-axis and the ordinates $x = \pi/4$ and $x = 3\pi/4$, then the area under the straight line $y = 1/\sqrt{2}$ between the x-axis and the ordinates $x = \pi/4$ and $x = 3\pi/4$, and then subtract the latter from the former.

The area under the curve is

$$\begin{aligned}
\int_{\pi/4}^{3\pi/4} y \, \mathrm{d}x &= \int_{\pi/4}^{3\pi/4} \sin x \, \mathrm{d}x \\
&= [-\cos x]_{\pi/4}^{3\pi/4} \\
&= \left[-\cos\frac{3\pi}{4}\right] - \left[-\cos\frac{\pi}{4}\right] \\
&= \left[-\left(-\frac{1}{\sqrt{2}}\right)\right] - \left[-\frac{1}{\sqrt{2}}\right] \\
&= \frac{2}{\sqrt{2}},
\end{aligned}$$

and the area under the straight line is the area of a rectangle, which is its length times its height, giving

$$\begin{aligned}
\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) \times \frac{1}{\sqrt{2}} &= \frac{\pi}{2} \times \frac{1}{\sqrt{2}} \\
&= \frac{\pi}{2\sqrt{2}}.
\end{aligned}$$

Thus the required area is $2/\sqrt{2} - \pi/2\sqrt{2} = (4 - \pi)/2\sqrt{2}$.

Exercises

5. Calculate the area contained between the curve $y = \cos x$, $-\pi < x < \pi$, and the line $y = \frac{1}{2}$.
6. Find the area contained between the line $y = x$ and the curve $y = x^2$.
7. Find the area contained between the two curves $y = 3x - x^2$ and $y = x + x^2$.

Answers

1. (a) 4 (b) $20\frac{2}{3}$ (c) $10\frac{2}{3}$ (d) $\frac{2}{3}$
(note that the integrals for parts (c) and (d) are negative, but the areas are positive).

2. $\frac{1}{2}$

3. 0. The areas between $x = -1$ and $x = 0$, and between $x = 0$ and $x = 1$, are equal in size. However, one area is above the axis and one is below the axis, so that the integrals have opposite sign.

4. 0. The curve crosses the x-axis at $x = 0$ and $x = 4$. The areas between $x = 0$ and $x = 4$, and between $x = 4$ and $x = 6$, are equal in size. However, one area is above the axis and one is below the axis, so that the integrals have opposite sign.

5. $\sqrt{3} - \pi/3$

6. $\frac{1}{6}$

7. $\frac{1}{3}$