Area under a Curve

Calculating the area under a curve.

Definite Integrals

So far when integrating, there has always been a constant term left. For this reason, such integrals are known as indefinite integrals. With definite integrals, we integrate a function between 2 points, and so we can find the precise value of the integral and there is no need for any unknown constant terms [the constant cancels out].

$$\int_4^8 x + 3x^2 \, dx$$

This means the integral of $x + 3x^2$ with respect to x, between x=8 and x=4.

Integrating:

$$\left[\frac{x^2}{2} + \frac{3x^3}{3} + c\right]_4^8$$

$$= \left(\frac{8^2}{2} + \frac{3(8)^3}{3} + c\right) - \left(\frac{4^2}{2} + \frac{3(4)^3}{3} + c\right)$$

$= (32 + 512 + c) - (8 + 64 + c)$

$= 544 + c - 72 - c$

$= \underline{\underline{472}}$

Note that the constant terms disappear, and always will with definite integrals so you don't have to write them in.

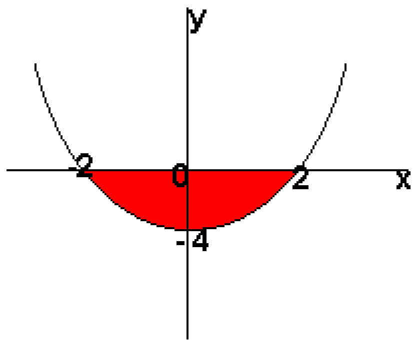
The Area Under a Curve

The area under a curve between two points can be found by doing a definite integral between the two points.

To find the area under the curve y = f(x) between x = a and x = b, integrate y = f(x) between the limits of a and b.

Example 1

Example: What is the area between the curve $y = x^2 - 4$ and the x axis?



The shaded area is the area that we want.

We can easily work out that the curve crosses the x axis when x = - 2 and x = 2. To find the area, therefore, we integrate the function between -2 and 2.

$$\int_{-2}^{2} (x^2 - 4)\,dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2} = \left(\frac{8}{3} - 8\right) - \left(\frac{-8}{3} + 8\right)$$

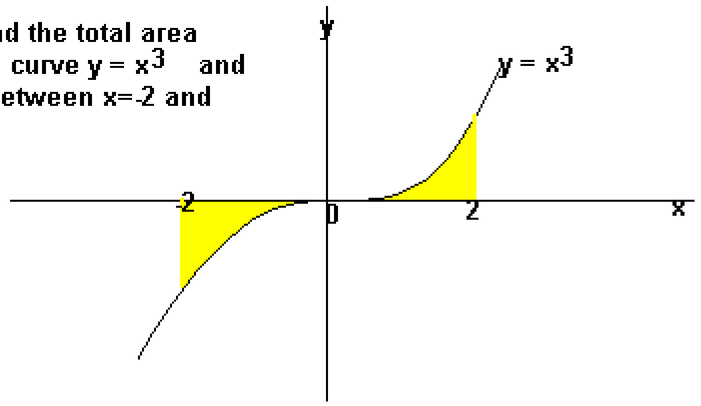
$= 16/3 - 16$
$= \underline{-10.67}$ (2d.p.)

Note: the area is negative because it is below the x-axis. Areas above the x-axis, on the other hand, give positive results.

Example 2

Areas under the x-axis will come out negative and areas above the x-axis will be positive. This means that you have to be careful when finding an area which is partly above and partly below the x-axis.

Example: find the total area between the curve $y = x^3$ and the x-axis between x=-2 and x=2.



If we simply integrated x^3 between -2 and 2, we would get:

$$\left[\frac{x^4}{4}\right]_{-2}^{2} = 4 - 4 = 0$$

So instead, we have to split the graph up and do two separate integrals.

$$\int_{0}^{2} x^3\,dx = \left[\frac{x^4}{4}\right]_{0}^{2} = 16/4 - 0 = 4$$

$$\int_{-2}^{0} x^3\,dx = \left[\frac{x^4}{4}\right]_{-2}^{0} = 0 - 16/4 = -4 \quad \text{(so area is 4).}$$

We then add these two up to get: $\underline{\underline{8 \text{ units}^2}}$

You may also be asked to find the area between the curve and the y-axis. To do this, integrate with respect to y.

Example 3

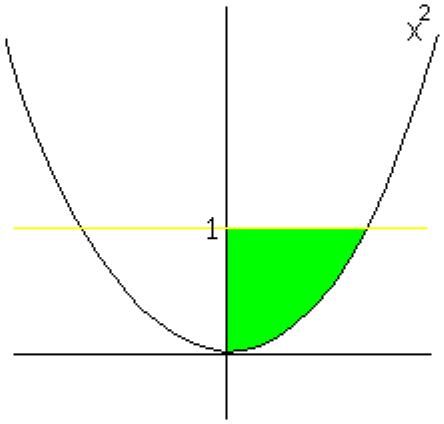
Example

Find the area bounded by the lines y = 0, y = 1 and y = x^2.

$$\text{Area} = \int_0^1 y^{1/2}\,dy \qquad \text{since } x = y^{1/2}$$

$$= \left[\frac{2y^{3/2}}{3}\right]_0^1$$

$$= 2/3$$

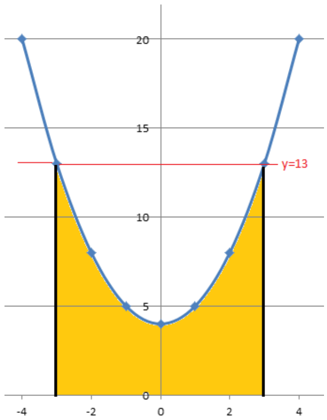


Exercise

Find the coordinates of the points (eg a and b) of intersection of the curve $f(x) = x^2 + 4$ and the line $y = 13$

Find the area bounded by the by $f(x)$, the x-axis and the lines x=a and x=b.

Solution



$$13 = x^2 + 4$$

$$13 - 4 = x^2$$

$$9 = x^2$$

$$\pm 3 = x \; or \; x = \pm 3$$

$\int_{-3}^{3} x^2 + 4$ = $\left|\frac{1}{3}x^3 + 4x + c\right|$ for $x = \pm 3$

Substituting we get

$\left|\frac{1}{3}(3)^3 + 4(3) + c\right| - \left|\frac{1}{3}(-3)^3 + 4(-3) + c\right|$

=$[9 + 12 + c]$—$9 - 12 + c = 21 + c + 21 - c$ (+c and –c cancel out)

Area = 42units^2