1 MODELLING USING MATHEMATICS

1.1 Explain the concept of a mathematical model to include the difference between mathematical models and physical models.

1.2 Explain the modelling process in diagrammatic form.

1.3 Solve simple mathematical models to include identifying situations requiring mathematical modelling, and using appropriate mathematical skills and processes.

1.4 Apply simple mathematical models to explain and predict behaviour.

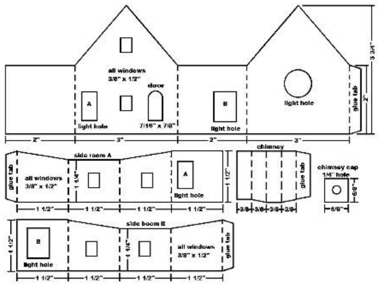
Physical Models

Physical models are three-dimensional representations of reality. In general, two types of physical models exist:

- mock-up
- prototypes

Mock-up

This type of physical model is an appearance, conceptual or physical mock-up. It is used to evaluate the styling, balance, colour, or other aesthetic feature of a technology artifact. Mock-ups are generally constructed of materials that are easy to work with. Commonly these materials include wood, clay, paper, and various kinds of cardboard.



Prototype

A prototype is a physical model which is also a working model of a system, assembly, or a product. Prototypes are constructed to test the operation, maintenance and safety of an item. Prototypes are generally constructed from the same material as the final product. Prototypes can be used to test and evaluate the solutions.

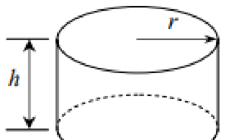
Mathematical Models

Mathematical models are comprised of numbers, mathematical formulae, equations, etc and used to provide a solution to a Real-World problem (eg cost of painting a room with a wall surface area of 60m^2).

Examples of Physical and Mathematical Models:

Surface area and volume

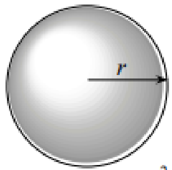
Cylinder



Curved surface area = $2\pi rh$

Volume = $\pi r^2 h$

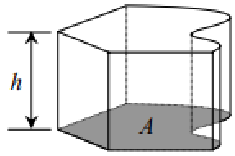
Sphere



Surface area = $4\pi r^2$

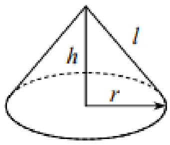
Volume = $\frac{4}{3}\pi r^2$

Solid of uniform cross-section (prism)



Volume = Ah, where A is the area of the base.

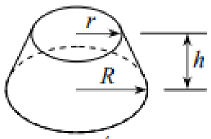
Cone



Curved surface area = πrl

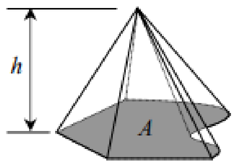
Volume = $\frac{1}{3}\pi r^2 h$

Frustum of a cone



Volume = $\frac{1}{3}\pi h\left(R^2 + Rr + r^2\right)$

Pyramid on any base



Volume = $\frac{1}{3}Ah$, where A is the area of the base.

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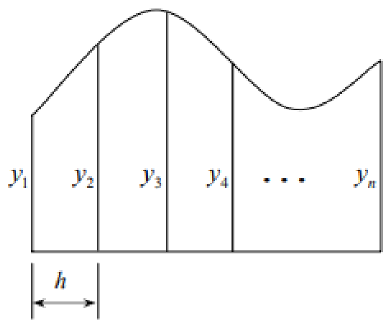
Physical Model - Real world challenge to calculate the area of a field (eg non standard geometric shape as shown below)

Mathematical Model – this is the maths formula as shown.

Area approximations



Trapezoidal rule:

Area $\approx \frac{h}{2}\left[y_1 + y_n + 2\left(y_2 + y_3 + y_4 + \cdots + y_{n-1}\right)\right]$

or $\approx \frac{h}{2}$[first + last + twice the rest]

Simpson's rule:

Area $\approx \frac{h}{3}\left[y_1 + y_n + 2\left(y_3 + y_5 + \cdots + y_{n-2}\right) + 4\left(y_2 + y_4 + \cdots + y_{n-1}\right)\right]$, (where n is odd)

or $\approx \frac{h}{3}$[first + last + twice the odds + four times the evens]

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Mathematical Modelling

Method of simulating real-life situations with mathematical equations to forecast their future behaviour. Mathematical modelling uses tools such as decision-theory, queuing theory, and linear programming, and requires large amounts of number crunching.

- Mathematics is a very precise language which helps us formulate ideas.
- Mathematics is a concise language comprising of numbers and symbols, with well-defined rules (eg BIRDMAS, BOMDAS).
- Mathematical modelling is about using results that mathematicians have proved over hundreds of years.
- Computers can be used to perform numerical calculations.
  - (https://www.mathsisfun.com/data/function-grapher.php)
- Any formula/equation/etc can be considered a mathematical model.
  - (e.g. $y = mx + c$ where m is the slope of the line and c is the y-intercept in other words where the line cuts the y-axis)
- Linear Programming is a good example of where we use mathematics concepts, formula, etc to provide valuable information in real world scenarios.

Mathematical Modelling objectives

- Develop mathematical/scientific understanding from current knowledge of a system.
- Test the effect of changes in a system using various data (eg numbers, etc).
- Aid decision making, including:
  - tactical decisions made by managers,
  - strategic decisions made by planners.

Using Mathematical Modelling to Solve Real World Problems

Creating a mathematical model

- We begin with a word problem (eg real world problem).
- Determine what the objective is or what question we are to answer.
- Assign variables to quantities in the problem so you can calculate an answer using these variables (eg x, y)
- Derive mathematical equations/inequalities containing these variables.
- Use these equations/inequalities to find the values of these variables.
- Determine the answer to the problem.

In **Real World** situations we often make use of a **Model World** where models play an important role in planning and decision-making.

In mathematical modelling, we translate those beliefs into the language of mathematics. This has many advantages: • Mathematics is a very precise language. This helps us to formulate ideas and identify underlying assumptions. • Mathematics is a concise language, with well-defined rules for manipulations. • All the results that mathematicians have proved over hundreds of years are at our disposal. • Computers can be used to perform numerical calculations.	What objectives can modelling achieve? • Developing scientific understanding - through quantitative expression of current knowledge of a system (as well as displaying what we know, this may also identify what we do not know); • Test the effect of changes in a system; • Aid decision making, including: (i) tactical decisions by managers; (ii) strategic decisions by planners

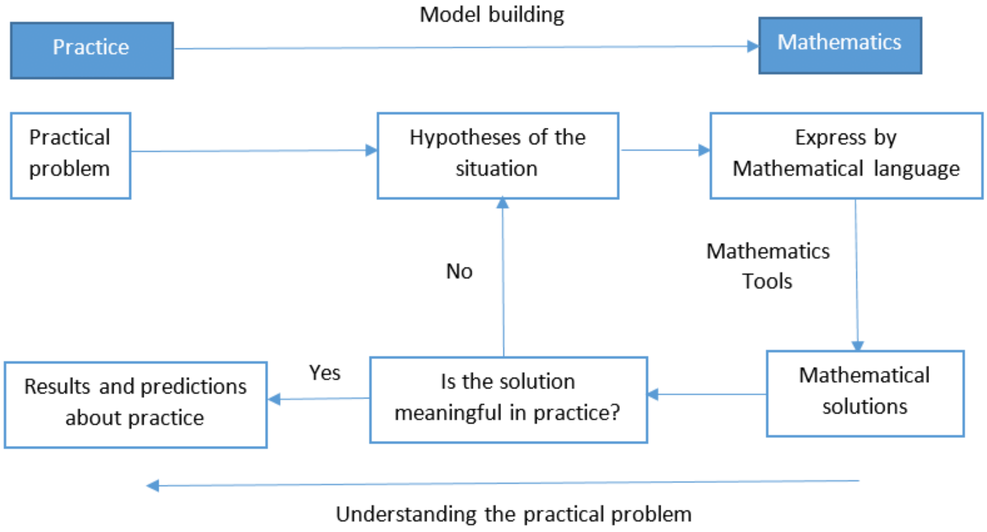
Benefits of using mathematical/physical models:

- Models help us communicate.
- Models allow us to clarify and test understanding.
- Models create credibility and accountability.
- Models help you organize your thoughts.
- Models simplify and solve problems.
- Models help you understand your data.

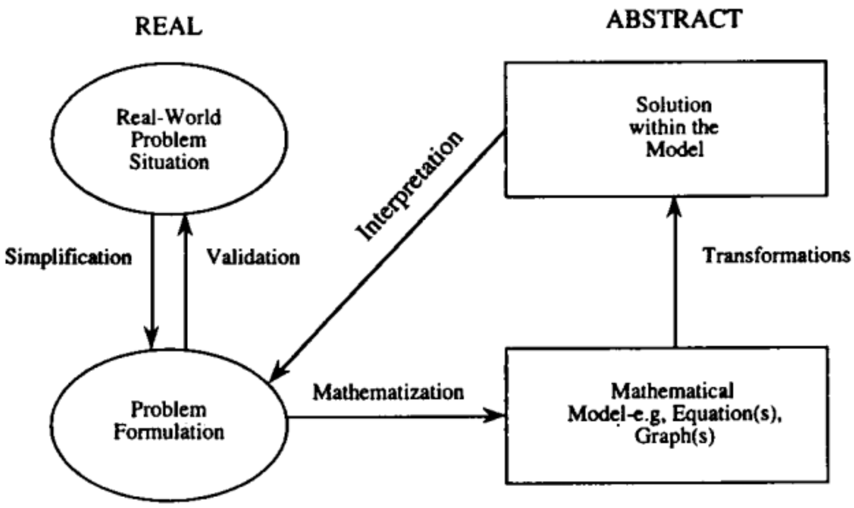
Mathematical Modelling Process in Diagrammatic form

Hypothesis - In mathematics, a hypothesis is an unproven statement which is supported by all the available data and results.

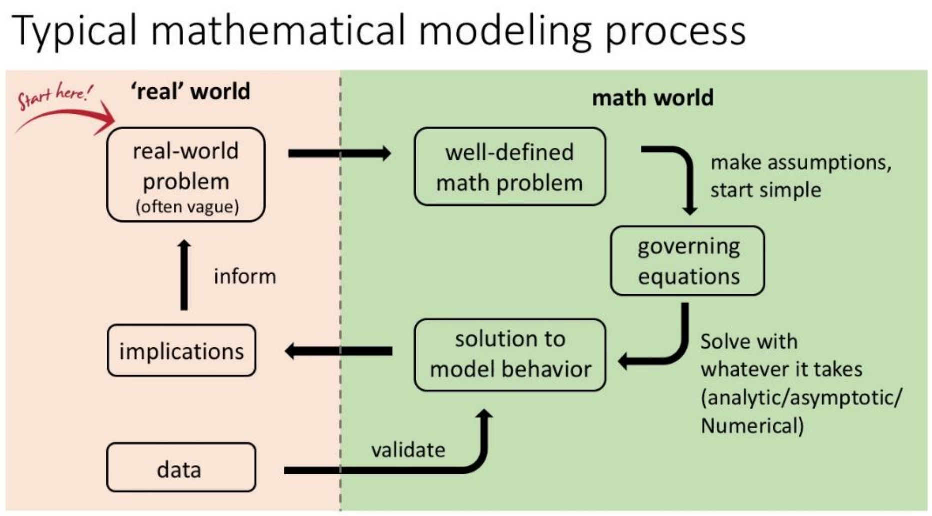
Example 1



Example 2

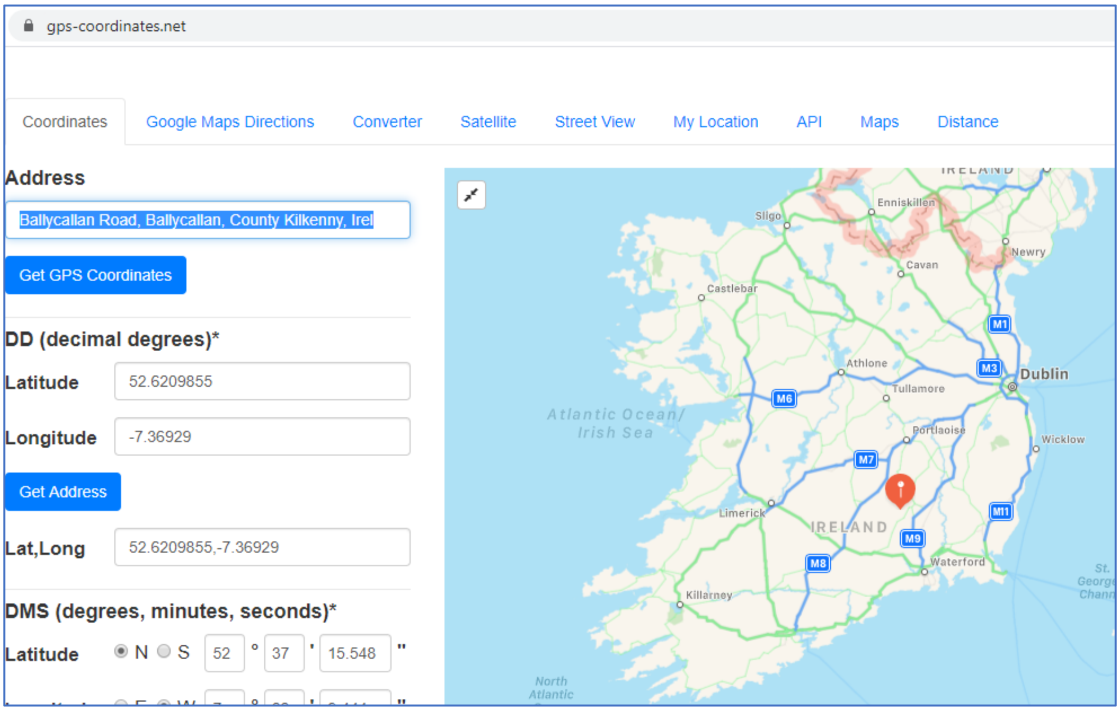


Example 3



Real World Challenge – Calculate the Distance between 2 locations

- Using our **Physical Model**: Determine the geolocation (eg GPS coordinates of each city or location).



- Use the **Mathematical Model** below to calculate the **distance** based on the GPS coordinates.

Co-ordinate geometry

Line

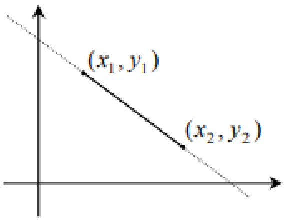
Slope: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

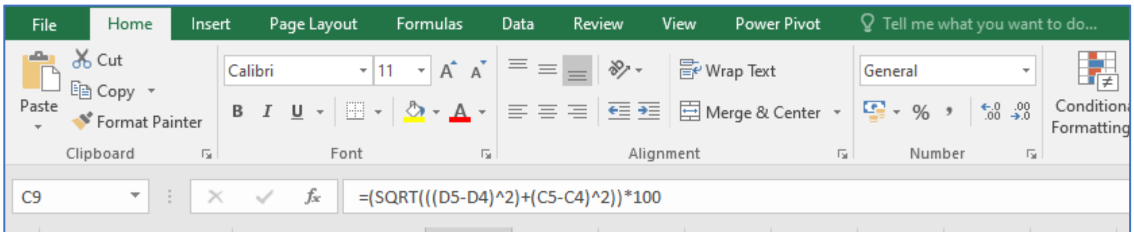
Distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Midpoint $= \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2} \right)$

Equation of a line: $y - y_1 = m(x - x_1)$ or $y = mx + c$

Area of a triangle with one vertex at the origin $= \frac{1}{2}|x_1 y_2 - x_2 y_1|$





	A	B	C	D	E
1	https://www.gps-coordinates.net/				
2					
3			Longitude	Latitude	
4	GPO, Dublin		-6.26749	53.3441	
5	GPO, Cork		-8.47109	51.89787	
6					
7			x	y	
8		Midpoint	-7.36929	52.62099	or Ballycallan Road, Ballycallan, County Kilkenny, Ireland
9		Distance	263.5799	km	
10		Slope			

- We can use pen/paper or calculator or software to calculate the correct answer to our challenge.

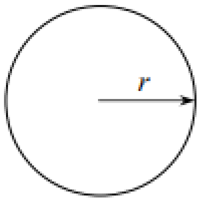
Solving Real World challenges using Mathematical Modelling

1. Example: 3 tennis balls(radius = 2cm) in a tube - how much plastic is required to make a simple tube(surface area required).

2. Example: 4 golf balls(radius = 12mm) in a cardboard box - how much cardboard is required to make the cardboard container(surface area required)

3. 8 dice (side = 3.25cm) in a container etc?? Answer will depend on the container design???

Circle / Disc

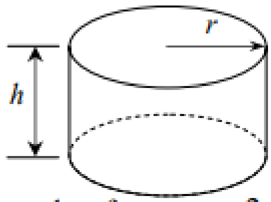


Length of circle = $2\pi r$

Area of disc = πr^2

Surface area and volume

Cylinder



Curved surface area = $2\pi rh$

Volume = $\pi r^2 h$

Solutions

Questions

Simple formulas as models

1. A builder charges €115 per sq. m. when building. Calculate the cost of building a house of 1,750 sq. m.
2. Calculate the total cooking time of a meal that weighs 0.842 kg. if the cooking instructions in your cookbook state:

   Cooking duration = (no. of kg.) * (minutes per kg.) + 10 minutes
3. What is the circumference of a circle of radius 3 m.
4. What is the area of a triangle of base 1.2 m. and height 3.8 m.

Conversion formulas as models

1. Next to the Elephant, the White Rhino is the largest land mammal and can weigh up to 3.6 metric tons. What weight is that in stones (1 stone = 14 lbs.)?
2. If Victor Costello weighs 18 ½ stone, how many kilograms (kg) does he weigh?
3. The local swimming pool contains 1,000 gallons of water. How many litres is that?
4. It's 101 km from Cork to Limerick. What is that in miles?
5. If it's 22 miles from Tralee to Killarney, how many kilometres is that?
6. A marathon is approximately 26.2 miles long. Convert that to kilometres.
7. Ronan O'Gara is 6ft tall exactly. What is his height in metres?
8. Carrauntoohill is 3,414 ft in height. Convert that to metres.
9. The Croke park pitch is 100 by 150 yards. What size is that in metres^2?
10. How many acres in a 15.3 ha. farm?
11. How many metres squared is there 6 acres?
12. If it 18 degrees Celsius today, what is the temperature in Fahrenheit?

Growth models

Calculate the compound interest on each of the following (when necessary, give your answers correct to the nearest cent):

1. €350 invested for 3 years at 10% per annum.
2. €2,500 invested for 3 years at 6% per annum.
3. €6,500 was invested for three years at compound interest. The interest rate for the first year was 5%, for the second year 8% and 12% for the third year. Calculate the total interest earned.
4. A person invests €2,000 at 14% compound interest and withdraws it in two equal annual instalments beginning one year from the date on which it was invested. Find the value of each instalment correct to the nearest cent. (Hint: Let x = one of the instalments.)
5. A sum of money, invested at compound interest, amounts to €7,581.60 after two years at 8% per annum. Calculate the sum invested.
6. Assume bacteria multiply at a rate of 8% per hour, when in a suitable environment. If left unchecked, how many bacteria would you expect to get after 3 days if there were 1,000 bacteria to begin with? We are assuming unlimited resources available and no mortality.

Solutions to Exercises at end of “Maths-Modelling-Diagrammatic-Form”

Simple Formula as Models

1. 115(1750) = €201,250
2. Assuming it takes 10 minutes per kg to cook (0.842)(10) + 10 = 18.42 minutes
3. Circumference $C = 2\pi r$, $2(\pi)(3) = 6\pi$ or $18.857142857 \ldots. m$
4. Area = ½ Base by Perpendicular Height, ½ (1.2)(3.8) = 2.28m^2

Conversion Formula as Models

1. Weight by Conversion Factor (1 ton = 157.47 stone), (3.6)(157.47) = 566.892 stone
2. Conversion Factor (1 stone = 6.35029kg), (18.5)(6.35029) = 117. 480365kg
3. Conversion Factor (1gallon = 3.78541litres), (1000)(3.78541) = 3785.41litres
4. Conversion Factor (1km = 0.6214miles), (101)(0.6214) = 62.7614 miles
5. Conversion Factor (1mile = 1.60934km), (22)(1.60934) = 35.40548km

Growth Models

1.	$A = P(1 + \frac{r}{100})^n$, $A = 350(1 + \frac{10}{100})^3$ = $A = 350(1.1)^3$ = $350(1.1331)$ = €465.85
2.	$A = P(1 + \frac{r}{100})^n$, $A = 2500(1 + \frac{6}{100})^3$ = $A = 2500(1.06)^3$ = $2500(1.191016)$ = €2977.54

3.

€6500

Year 1 : $6500\left(\frac{105}{100}\right) = 6500(1.05) = 6825$

Year 2 : $6825\left(\frac{108}{100}\right) = 6825(1.08) = 7371$

Year 3 : $7371\left(\frac{112}{100}\right) = 7371(1.12) = 8255.52$

$$\begin{array}{r} 8255.52 \\ -\ 6500.00 \\ \hline €1755.52 \end{array}$$

← Total interest earned

Maths Model

Year 1 : $1.05x$

Year 2 : $(1.05x)(1.08) = 1.134x$

Year 3 : $(1.134x)(1.12) = 1.27008x$

$\boxed{\text{Mathematical Model} = 1.27008x}$

Sample Excercises

1 Let $x = 6500$ $\Rightarrow$ $6500(1.27008)$

2 Let $x = 5000$

3 Let $x = 1000$

4.

$$2000\left(1+\frac{14}{100}\right)$$

$$((2000(1.14)-x)(1.14))-x=0$$

$$((2280-x)1.14)-x=0$$

$$(2599.20-1.14x)-x=0$$

$$2599.20-2.14x=0$$

$$2599.20=2.14x$$

$$\frac{2599.20}{2.14}=x$$

$$1{,}214.579439252336=x$$

$$1{,}214.58=x$$

5. $7581.60 = x(1+\frac{8}{100})^2$ = 7581.60= $x(1.08)^2$ = $\frac{7581.60}{1.1664}$=x therefore x= 6500

6.

$$A = 1000\left(1 + \frac{8}{100}\right) \qquad 3 \times 24 = 72 \text{ terms}$$

$$= 1000\left(1.08\right)^{72}$$

$$= 1000\left(254.982511844\right.$$

$$= 254{,}982.511844$$

Mathematical Models to explain and predict behaviour.

Maths and Science have many kinds of models:

- Animal Models: - To study human diseases.
- Physical Models: - To demonstrate the apparent truth of a principle.
- Mathematical Models: - To gain insight and understanding of a phenomenon which is being observed.

Inputs ⇒ MODEL ⇒ Outputs

Models can be used to:

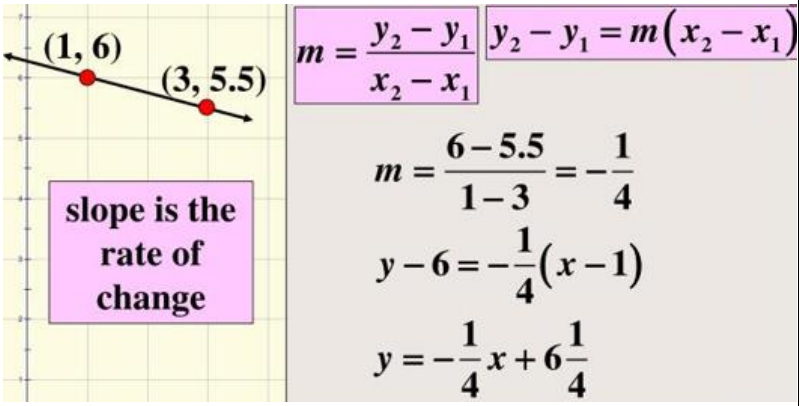
- Understand the problem/challenge/phenomenon better.
- Make predictions of how outcome change with inputs.

Use graphical models to make predictions or assist in decision making.

Example	Description
1	**Aim**: How do we use data to make predictions – (eg linear regression)?

Scenario: A candle is 6cm tall after burning for 1 hour. After 3 hours, it is 5.5cm tall. Write a linear equation to model the height y of the candle after burning for x hours.

Mathematical Model:



Question: How tall would the candle be after burning for 4 hours

$$y = -\frac{1}{4}(4) + 6\frac{1}{4} \qquad = -1 + 6\frac{1}{4} \quad = 5\frac{1}{4} cm \ or \ 5.25cm$$

It was easy to make a prediction as to how tall the candle would be using the mathematical model.

Example	Description
2	The table shows the temperature of a fish tank during an experiment. Write the appropriate linear equation for the given situation and use the equation to predict the temperature at the 7th hour.

Time (h)	0	1	2	3	4	5
Temperature (°F)	82	80	78	76	74	72

Solution :

Step 1 :
Notice that the change in the temperature is the same for each increase of 1 hour in time. So, the relationship is linear.

	Since we want to find the temperature at the 7th hour, the appropriate linear equation for the given situation is slope-intercept form (y = mx + b), assuming "y" as temperature and "x" as hours. **Step 2 :** Choose any two points in the form (x, y), from the table to find the slope : For example, let us choose (0, 82) and (1, 80). Use the slope formula. $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{80-82}{1-0} = \frac{-2}{1} = -2$ **Step 3 :** Find the y-intercept(b) using the slope and any point from the table. Slope-intercept form equation of a line : y = mx + b Substitute m = -2, and (x, y) = (0, 82). 82 = -2(0) + b 82 = 0 + b 82 = b **Step 4 :** Now, substitute m = -2 and b = 82 in slope-intercept form equation of a line. y = mx + b y = -2x + 82 **Step 5 :** Predict the temperature at the 7th hour. Substitute x = 7 in the equation y = -2x + 82. y = -2(7) + 82 y = -14 + 82 y = 68 So, the temperature at the 7th hour is 68^{o} F.
3	Elizabeth's cell phone plan lets her choose how many minutes are included each month. The table shows the plan's monthly cost y for a given number of included minutes x. Write an equation in slope-intercept form to represent the situation and use it to predict cost of plan for 800 minutes included. <table><tr><td>**Minutes included, *x***</td><td>100</td><td>200</td><td>300</td><td>400</td><td>500</td></tr><tr><td>**Cost of plan (€), *y***</td><td>14</td><td>20</td><td>26</td><td>32</td><td>38</td></tr></table> Solution : Step 1 : Notice that the change in cost is the same for each increase of 100 minutes. So, the relationship is linear. Step 2 : Choose any two points in the form (x, y), from the table to find the slope : For example, let us choose (100, 14) and (200, 20). Use the slope formula and substitute values for y_2,y_1,etc : . $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{20 - 14}{200 - 100} = \frac{6}{100} = 0.06$

	Step 3 : Find the y-intercept using the slope and any point from the table. Slope-intercept form equation of a line : $y = mx + b$ Substitute $m = 0.06$, and $(x, y) = (100, 14)$. $14 = 0.06(100) + b$ $14 = 6 + b$ $8 = b$ Step 4 : Now, substitute $m = 0.06$ and $b = 8$ in slope-intercept form equation of a line. $y = mx + b$ $y = 0.06x + 8$ Step 5 : Predict cost of plan for 800 minutes included. Substitute $x = 800$ in the equation $y = 0.06x + 8$. $y = 0.06(800) + 8$ $y = 48 + 8$ $y = 56$ So, the cost of plan for 800 minutes included is €56.
4	The rent charged for space in an office building is a linear relationship related to the size of the space rented. At West Main Street office rentals, €750 rent charged for 600 square feet of space and €1150 rent charged for 900 square feet of space. Write an equation in slope-intercept form for the rent at West Main Street Office Rentals and use it to predict the rent for 1200 square feet of space. **Solution :** **Step 1 :** Identify the independent and dependent variables. The independent variable (x) is the square footage of floor space. The dependent variable (y) is the monthly rent. **Step 2 :** Write the information given in the problem as ordered pairs. The rent for 600 square feet of floor space is \$750 : (600, 750) The rent for 900 square feet of floor space is \$1150 : (900, 1150)

Step 3 :

Find the slope by substituting values as shown:

$$(x_1, y_1) = (600, 750)$$
$$(x_2, y_2) = (900, 1150)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1150 - 750}{900 - 600} = \frac{400}{300} = \frac{4}{3} = 0.06$$

Substitute :

Step 4 :

Find the y-intercept.

Use the slope $\frac{4}{3}$ and one of the ordered pairs (600, 750).

Slope-intercept form :

$$y = mx + b$$

Substitute m = 4/3, x = 600 and y = 750.

$$750 = \tfrac{4}{3}(600) + b$$
$$750 = 800 + b$$
$$-50 = b$$

Step 5 :

Substitute the slope and y-intercept.

Slope-intercept form

$$y = mx + b$$

Substitute m = $\frac{4}{3}$ and b = -50.

$$y = (\tfrac{4}{3})x + (-50)$$
$$y = (\tfrac{4}{3})x - 50$$

Step 6 :

Predict the rent for 1200 square feet of space.

Substitute x = 1200 in the equation y = $(\frac{4}{3})$x - 50.

$$y = (\tfrac{4}{3})(1200) - 50$$
$$y = 1600 - 50$$
$$y = 1550$$

So, the rent for 1200 square feet of space is €1550.

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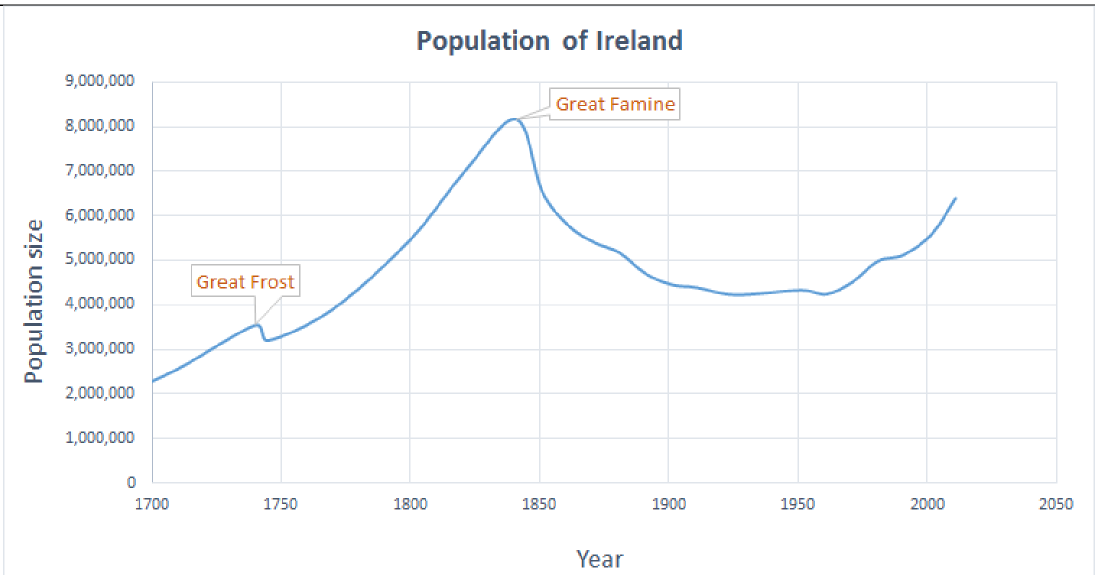

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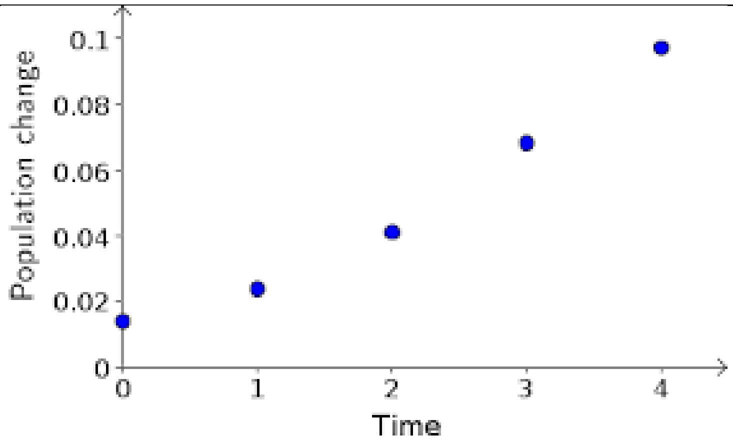
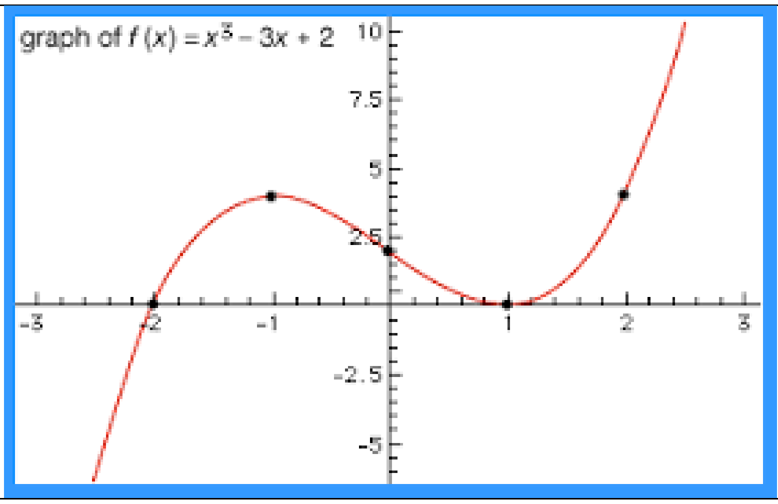
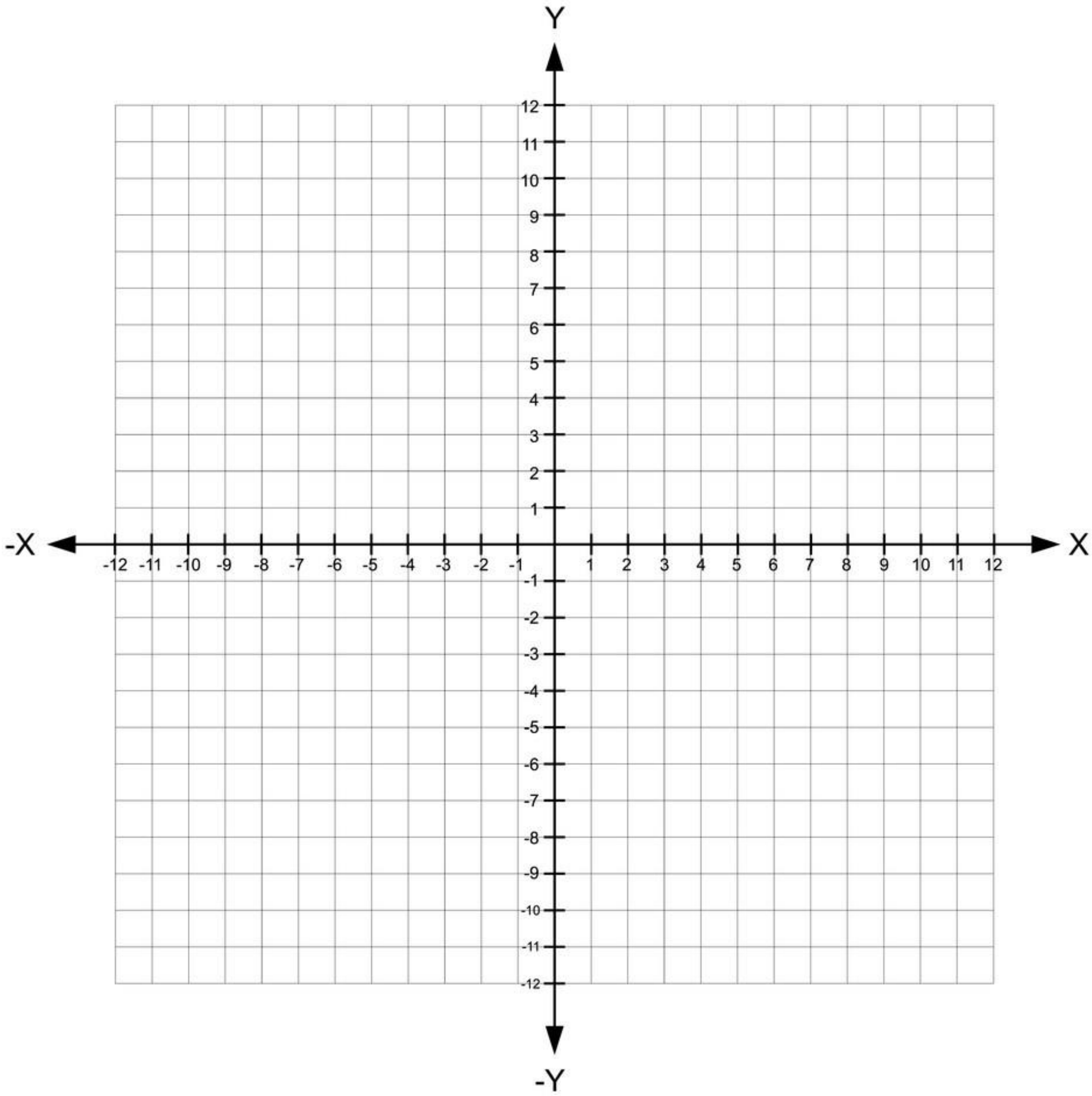
Bitcoin: Stock to Flow Ratio





Coordinate Geometry – Line

Cartesian Plane *(names after the French mathematician De Cartes)*



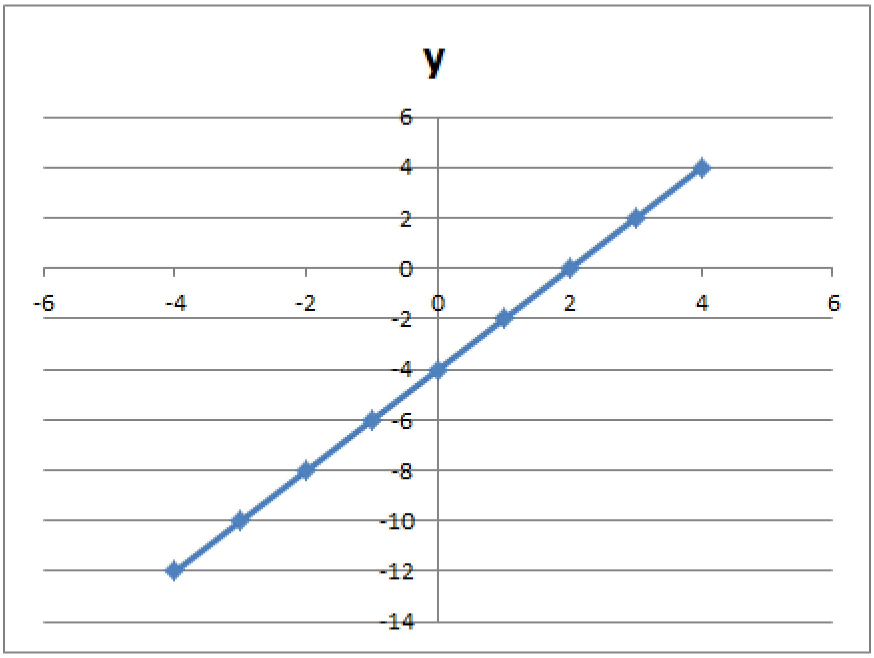
Plot the following points on the cartesian plane:

1. (3,6)
2. (-2,8)
3. (-4,-2)
4. (6,-6)

Graphing lines

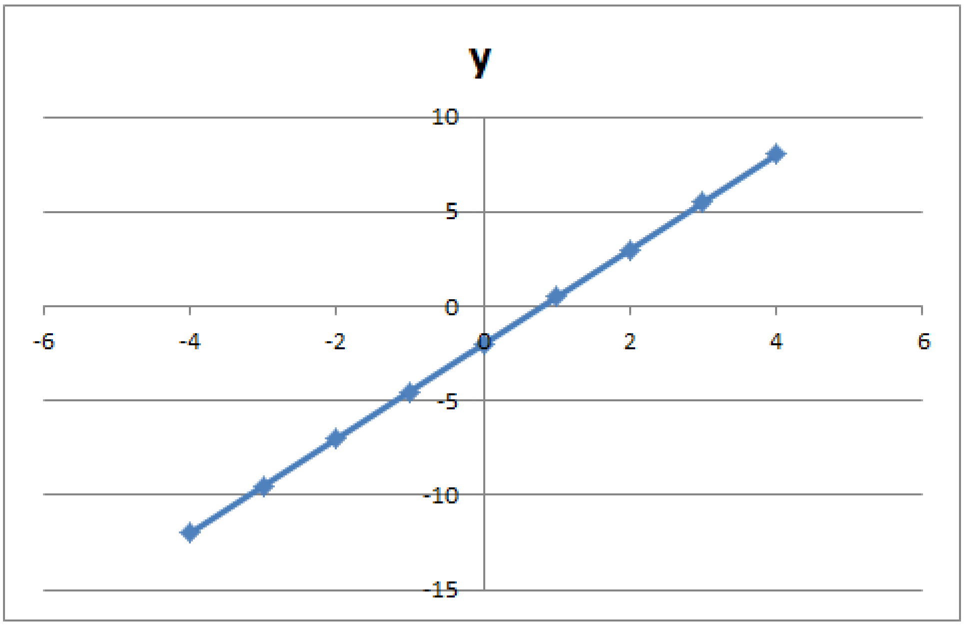
y=2x-4

y = 2x - 4	x	-4	-3	-2	-1	0	1	2	3	4
	2x	-8	-6	-4	-2	0	2	4	6	8
	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
	y	-12	-10	-8	-6	-4	-2	0	2	4



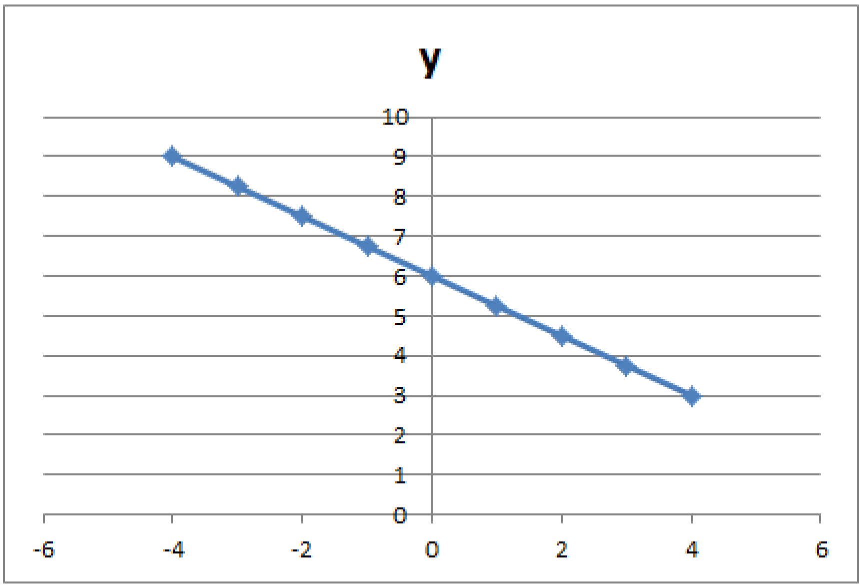
5x-2y=6

5x-2y =6	x	-4	-3	-2	-1	0	1	2	3	4
=> y=(5x-6)/2	5x/2	-10	-7.5	-5	-2.5	0	2.5	5	7.5	10
or y=5x/2 -6/3	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
=> y=5x/2-2	y	-12	-9.5	-7	-4.5	-2	0.5	3	5.5	8



3x+4y-12=0

3x+4y-12=0	x	-4	-3	-2	-1	0	1	2	3	4
=> y=-3/4x+3	-3/4x+3	6	5.25	4.5	3.75	3	2.25	1.5	0.75	0
	3	3	3	3	3	3	3	3	3	3
	y	9	8.25	7.5	6.75	6	5.25	4.5	3.75	3



Formula	
Midpoint of a line segment	$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
Equation of a line	$y - y_1 = m(x - x_1)$
Slope of a line	$y = mx + c$
Slope of a line given points on line	$m = \frac{y_2 - y_1}{x_2 - x_1}$
Length of line segment or distance between 2 points	$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Parallel lines have the same slope	$m_1 = m_2$
Perpendicular lines slopes multiply to give -1	$m_1 . m_2 = -1$

Exercises

Find the Mid-point, Slope, Length of line segment and Equation of the lines for the following points:

- $(-1, 1)$ and $(3, 4)$
- $(1, 2)$ and $(-2, 6)$

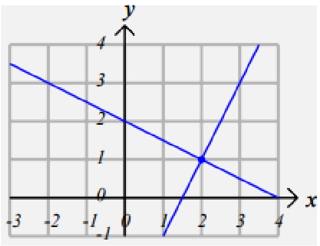
Perpendicular Lines

Two lines with slopes m_1 and m_2 are parallel if $m_1 = m_2$.

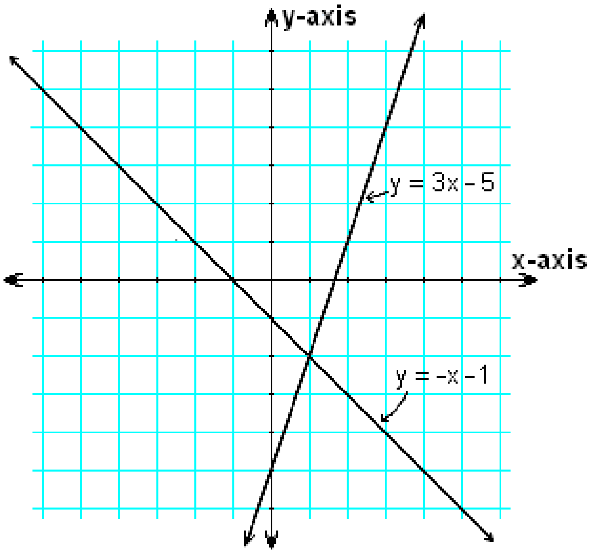
perpendicular if $m_1 m_2 = -1$.

Example: Find the equation of the line through the point $(2, 1)$ which is perpendicular to the line $y = -\frac{1}{2}x + 2$.

Solution: The slope of the perpendicular line is $-\frac{1}{-\frac{1}{2}} = 2$, so the equation of the perpendicular line is, using the Point-Slope Form: $y - 1 = 2(x - 2)$



Intersecting Lines – Simultaneous Equations



$$y = 3x - 5$$
$$y = -x - 1$$

Cancel either the x's or y's. In this example we will cancel the y's so to do this we must multiply the bottom equation by -1.

$$y = 3x - 5$$
$$-y = +x + 1$$

A -y cancels with a +y, 3x+x=4x, etc

$$0 = 4x - 4 \; therefore \; 4x = 4 \; so \; x = \frac{4}{4} \; and \; x = 1$$

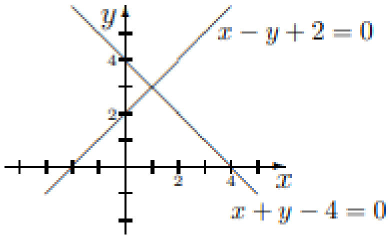
X=1 so we can substitute this into either equation

$$y = 3(1) - 5$$
$$y = 3 - 5$$
$$y = -2$$

Intersection point is

$$(1, -2)$$

Solving Simultaneous Equations



$$x + y - 4 = 0$$

$$x - y + 2 = 0$$

$$y + y - 4 = 0$$

$$x - y + 2 = 0$$

Using Linear Programming to solve simple mathematical models and mathematical modelling

Today we will do this using straight lines as our equations, and we will solve the problem by drawing these lines (graphing).

This process is called "Linear Programming" and is one of the most powerful mathematical methods used by businesses and companies to solve problems and help them make the best decisions.

"Operations Research" is the profession that applies mathematical methods like this to problems arising in industry, healthcare, finance, etc.

Test Questions 17B

1. Using the same axes and the same scales, indicate the region which simultaneously satisfies the inequalities
$$x \geq 0,\ y \geq 0 \text{ and } x + 2y \leq 8.$$

2. Shade in the set of couples which simultaneously satisfies these inequalities:
$$x \geq 1,\ y \geq 0 \text{ and } 5x + 2y - 10 \leq 0.$$

3. Illustrate the set of points which simultaneously satisfies the inequalities
$$x \geq 0,\ y \geq 1;\ 2x + y \leq 8 \text{ and } 2x + 3y \leq 12.$$

4. Shade in the set of couples (x, y) which simultaneously satisfies these inequalities:
$$y \geq 0;\ 2x - 3y \geq 0 \text{ and } 3x + 5y \leq 15.$$

5. Using the same axes and the same scales indicate the region which simultaneously satisfies the inequalities
$$x \geq 0,\ y \geq 0,\ x + 2y \leq 8 \text{ and } 3x + 2y \leq 12.$$

6. Using the same axes and the same scales plot the inequalities
$$x \geq 0,\ x + 2y \leq 4 \text{ and } x - y - 2 \leq 0.$$
Shade in the region that satisfies all three inequalities simultaneously.

7. Using the same axes and the same scales indicate the region which simultaneously satisfies the inequalities
$$y \geq 0,\ 3x - 2y \geq 0 \text{ and } 3x + 4y - 18 \leq 0.$$
Find the coordinates of the three vertices of this region.

8. Indicate the region which simultaneously satisfies these inequalities:
$$x \geq 1,\ y \geq 0;\ x + y - 6 \leq 0 \text{ and } 3x + y - 12 \leq 0.$$
Now find the coordinates of the four vertices of this region.

2. Shade in the region that simultaneously satisfies the inequalities

$$x \geq 0,\ x - 4y \leq 0 \text{ and } x + y \leq 5.$$

Find (i) the point (x, y) of this region which gives $3x - y$ its maximum value
(ii) the point (x, y) which gives $3x - y$ its minimum value.

3. Using the same axes and the same scales graph the following inequalities:

$$x \geq 0,\ y \geq 0,\ x + 2y - 8 \leq 0 \text{ and } 3x + 2y - 12 \leq 0.$$

Shade in the set of points which simultaneously satisfies these inequalities.
Find a point (x, y) of this set which gives $2x + 3y$ a maximum value.

4. Shade in the region which simultaneously satisfies the following inequalities:

$$x \geq 0,\ y \geq 0,\ 2x + y \leq 8 \text{ and } x + 3y \leq 9.$$

Find the point (x, y) of this region which gives
(i) $2x + 4y$ a maximum value
(ii) $3x - y$ a minimum value.

5. Graph the inequalities A, B and C as follows:
A: $y \geq 0$; B: $6x - y \geq 0$ and C: $2x + 3y - 20 \leq 0$.

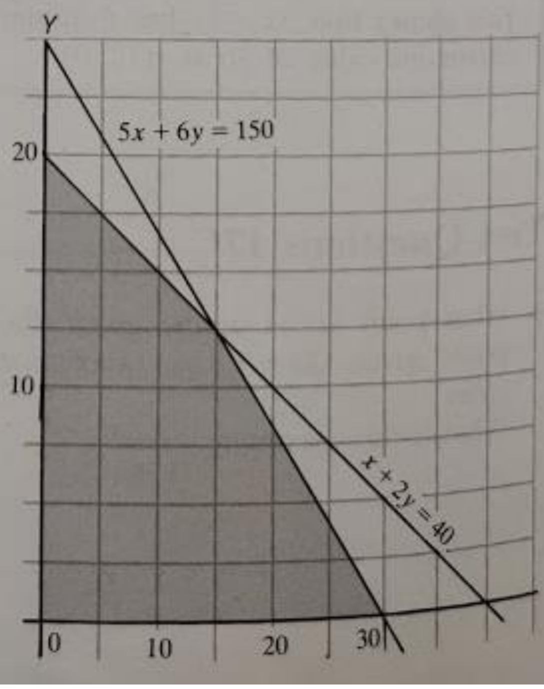
Shade in the region $A \cap B \cap C$.
Find the point (x, y) of this region which gives
(i) $3x + \frac{1}{2}y$ a maximum value
(ii) $5x - y$ a mimimum value.

6. Graph the following inequalities:
$x \geq 0$; $y \geq 0$; $8x + 12y \leq 96$ and $8x + 6y \leq 72$.

Shade in the region which simultaneously satisfies all four inequalities.
Find the four vertices of this region and hence find the point (x, y) which gives
(i) $4x + 5y$ a maximum value
(ii) $x + 3y$ a minimum value.

7. The point (x, y) lies within or on the boundary of the shaded area shown on the right.

Find (i) the maximum value
(ii) the minimum value
of $3x + y$.



Linear Programming Examples

Example 1

A shopkeeper can buy boxes of chocolates from a wholesaler at IR£10 per box and boxes of sweets at IR£8 per box. When buying his Christmas stock, the shopkeeper wishes to spend not more than IR£800 on these two items. To satisfy orders he has already taken, he needs to buy at least 20 boxes of chocolates and 25 boxes of sweets.

If the shopkeeper's profit is IR£4 on a box of chocolates and IR£2 on a box of sweets, how many of each should be purchased to maximize his profit. Find this profit.

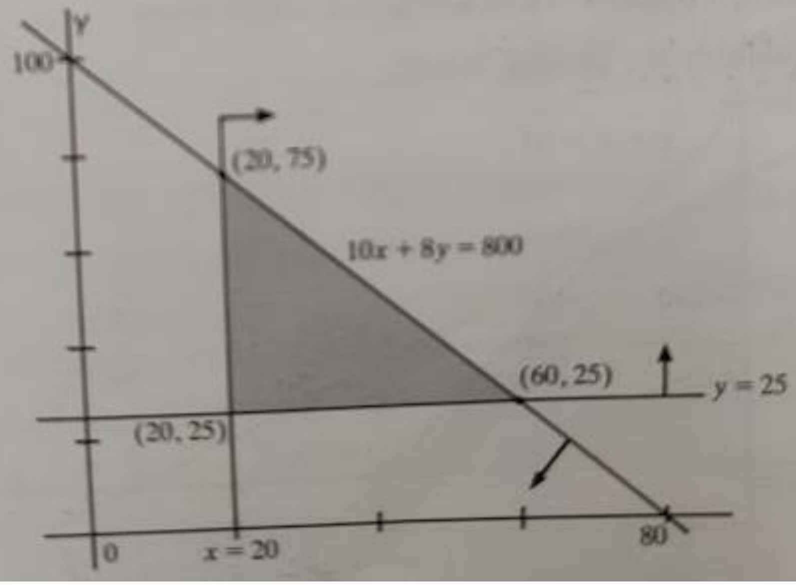
Step 1. Let x be the number of boxes of chocolates and y be the number of boxes of sweets.

Step 2. Write down the inequalities (or constraints) in x and y.

(a) Money constraint: $10x + 8y \leq 800$

(b) To satisfy orders already taken: $x \geq 20$ and $y \geq 25$.

Step 3: Plot these inequalities on a diagram.



The shaded area represents the intersection of the three inequalities.
The three vertices of this region are: $(20, 25)$, $(20, 75)$ and $(60, 25)$.

His profit = IR£$(4x + 2y)$.
We now find the value of $(4x + 2y)$ at each of the vertices.

Vertex	Value of $(4x + 2y)$
$(20, 25)$	$4(20) + 2(25) = 130$
$(20, 75)$	$4(20) + 2(75) = 230$
$(60, 25)$	$4(60) + 2(25) = 290$

So the shopkeeper needs to buy 60 boxes of chocolates and 25 boxes of sweets to maximize his profit. This profit is IR£290, as shown above.

Example 2

A builder is to build not more than 10 houses on an $8000\ \text{m}^2$ site. The houses are of two types—a terraced house which occupies $500\ \text{m}^2$ of ground and a detached house which occupies $1000\ \text{m}^2$ of ground.
Graph the set showing the possible numbers of each type of house that could be built.

The monthly rent he could obtain for the houses is as follows:

IR£400 for the terraced house
IR£600 for the detached house.

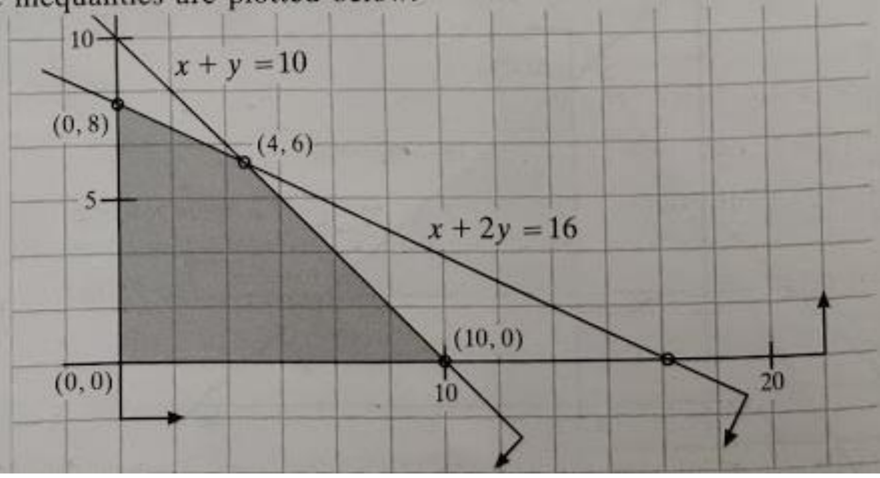
How many of each type should be built to give maximum rent?
Indicate on your graph the region where the rent would exceed IR£3000 per month.

Step 1: Let x be the number of terraced houses and y be the number of detached houses.

Step 2: Ground constraint: $500x + 1000y \leq 8000$.
i.e. $x + 2y \leq 16 \ldots$ (i)
Number constraint: $x + y \leq 10 \ldots$ (ii)

Also $x \geq 0$ and $y \geq 0$ (understood constraints)

These inequalities are plotted below:



The shaded region shows the possible numbers of each type of house that could be built.

The four vertices of this region are $(0, 0)$, $(10, 0)$, $(4, 6)$ and $(0, 8)$.

The rent is IR£$(400x + 600y)$.
We now find the value of $400x + 600y$ at each vertex.

Vertex	Value of $400x + 600y$
$(0, 0)$	0
$(10, 0)$	4000
$(4, 6)$	5200
$(0, 8)$	4800

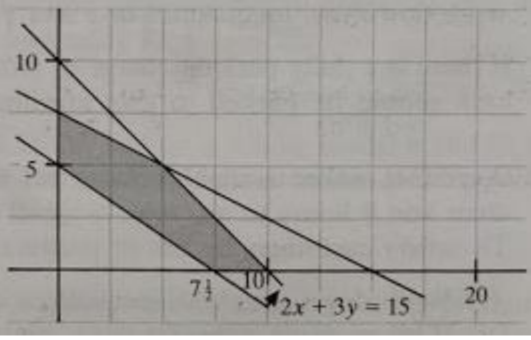
So 4 terraced houses (x) and 6 detached houses (y) should be built to maximize the rent. This maximum is IR£5200.

If the rent is to exceed IR£3000 per month, this gives the inequality

$$400x + 600y > 3000$$
$$\text{i.e. } 2x + 3y > 15$$

This inequality is graphed in the diagram below.

The shaded portion of the diagram on the right represents the region where the rent is greater than IR£3000.



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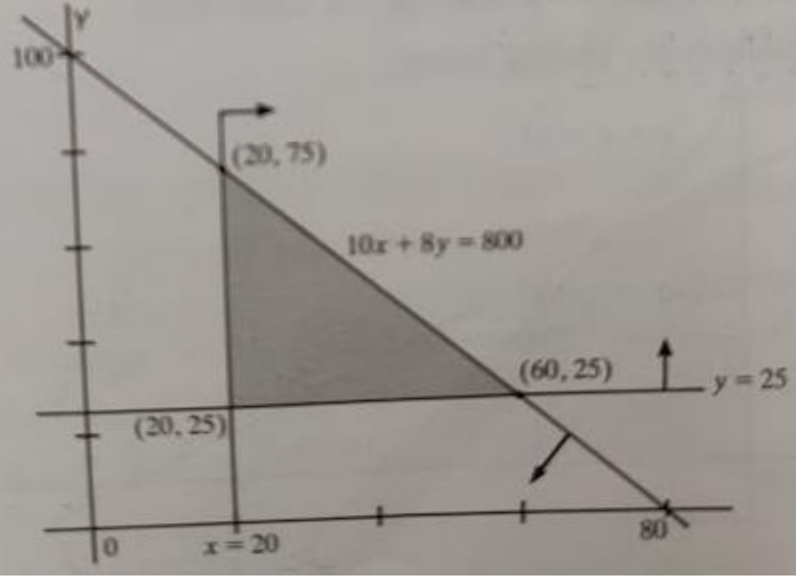

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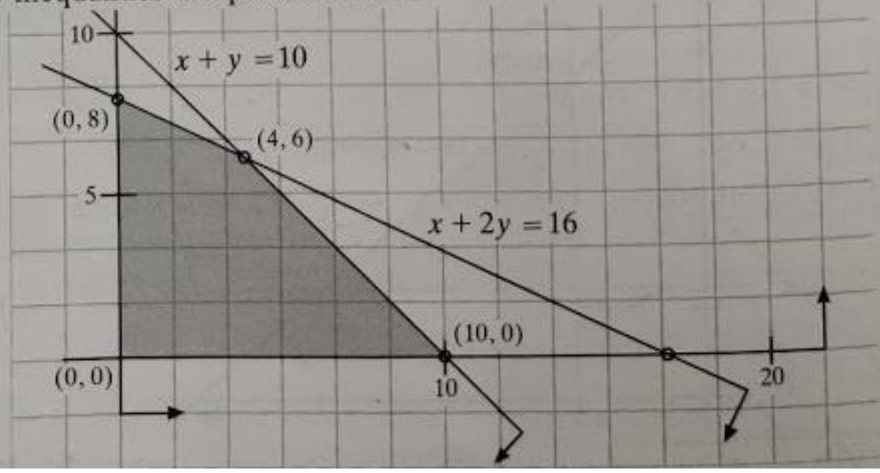

The shaded region shows the possible numbers of each type of house that could be built.

The four vertices of this region are $(0,0)$, $(10,0)$, $(4,6)$ and $(0,8)$.

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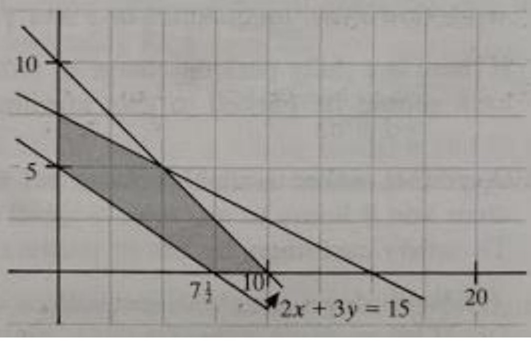


Linear Programming Exercises - Several

1. A manager plans to buy two types of machine, type A and type B, for his factory. He has 88 m^2 of space available for these machines. Type A needs 6 m^2 of floor space while type B needs 4 m^2 of floor space.
Type A costs IR£80 each and type B costs IR£400 each and the manager has up to IR£3600 to spend.
Assuming $x \geq 0$ and $y \geq 0$, write down two other inequalities in x and y.
Hence find the greatest number of machines he can buy.

2. A factory produces fridges and freezers. In any given week at most 200 of these products can be produced. Regular customers must be supplied with at least 20 fridges and 50 freezers per week.
Write down three inequalities to illustrate these constraints and then graph the inequalities.
If the profit on a fridge is IR£40 and the profit on a freezer is IR£50, how many of each should be produced to maximize profits.

3. A hotel carpark is 360 m^2. The parking area required for a car is 6 m^2 and for a bus 24 m^2. Not more than 30 vehicles could be accommodated at any given time. If x is the number of cars and y is the number of buses to be accommodated, write down two inequalities in x and y to illustrate these constraints.
If there is a daily parking charge of IR£1 per car and IR£3 per bus, how many of each should be parked to give maximum daily income, and state this income.

4. A cabinet maker assembles chairs and tables. It takes him 4 hours to assemble a chair and 8 hours to assemble a table, and he works 40 hours per week.
To satisfy customers he has to produce 2 chairs and 2 tables each week.
(i) Write down the three inequalities and illustrate them on a graph.
(ii) If his profit on a chair is IR£15 and his profit on a table is IR£35, how many of each should he assemble each week so as to maximize his profits?

5. A crystal glass factory produces two types of bowl, A and B. Each bowl A takes 3 hours on machine and 4 units of raw materials.
Each bowl B takes 2 hours on machine and 1 unit of raw materials.
In any week the factory has only 42 machine hours and 36 units of raw materials available.
If x of bowl A and y of bowl B are produced each week, write down two inequalities in x and y (other than $x \geq 0$ and $y \geq 0$).
(i) Illustrate these inequalities on a graph.
(ii) If the profit on each bowl A is IR£14 and on bowl B IR£9, calculate how many of each should be produced each week to maximize profits.

① A B

$x \geq 0$ $y \geq 0$

Constraints – Space

– Money

$$6x + 4y \leq 88 \longrightarrow 3x + 2y = 44$$

$$80x + 400y \leq 3600 \longrightarrow x + 5y = 45$$

$3x + 2y = 44$

<u>X-AXIS</u> $(y=0)$

$3x + 2(0) = 44$

$3x = 44$

$x = 14.\dot{6}$

$(14.6, 0)$

<u>Y-AXIS</u> $(x=0)$

$3(0) + 2y = 44$

$2y = 44$

$y = 22$

$(0, 22)$

$x + 5y = 45$

<u>X-AXIS</u> $(y=0)$

$x + 5(0) = 45$

$x = 45$

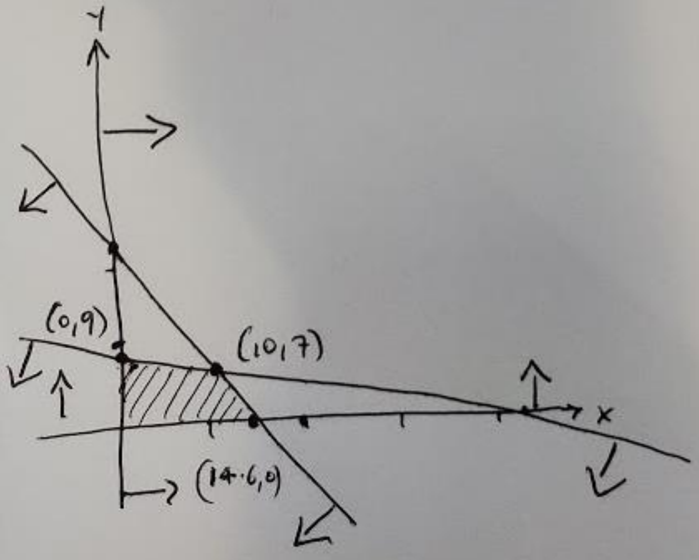
$(45, 0)$

<u>Y-AXIS</u> $(x=0)$

$(0) + 5y = 45$

$y = 9$

$(0, 9)$



$3x + 2y = 44$

$x + 5y = 45 \quad (x-3)$

$\cancel{3x} + 2y = 44$

$\cancel{-3x} - 15y = -135$

$-13y = -91$

$y = \frac{-91}{-13} = 7$

$x + 5(7) = 45$

$\therefore x = 10 \quad (10, 7)$

$\boxed{10 + 7 = 17 \text{ machines}}$

② Fridges Freezers

$x \geq 20$ $\quad$ $y \geq 50$ $\quad$ $x + y \leq 200$

$(0) + (0) \leq 200$ (True)

$x + y = 200$

x-Axis $(y = 0)$

$x + (0) = 200$

$x = 200$

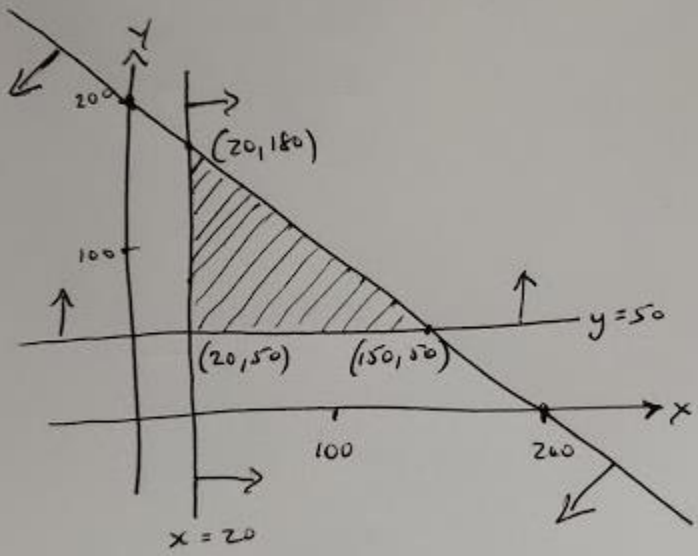
$(200, 0)$

Y-Axis $(x = 0)$

$(0) + y = 200$

$y = 200$

$(0, 200)$



$x = 20$ $\quad$ $(20) + y = 200$

$y = 180$ $\quad$ $(20, 180)$

$Y = 50$ $\quad$ $x + (50) = 200$

$x = 150$ $\quad$ $(150, 50)$

Profit $40x + 50y$

$40(20) + 50(50) = 3300$

$40(20) + 50(180) = \underline{9800}$

$40(150) + 50(50) = 8500$

Max Profit = £9,800

③ Car Bus

x y

Constraints — Area, number of vehicles

(i) $6x + 24y \le 360 \quad \div 6 \Rightarrow \boxed{x + 4y \le 60}$

(ii) $\boxed{x + y \le 30}$ $\quad \boxed{x \geq 0}$ $\quad \boxed{y \geq 0}$

Maximise profit $\Rightarrow \boxed{x + 3y}$

$x + y = 30$

X-Axis $(y=0)$

$x + (0) = 30$

$x = 30 \quad (30,0)$

Y-Axis $(x=0)$

$(0) + y = 30$

$y = 30 \quad (0,30)$

$x + 4y = 60$

X-Axis $(y=0)$

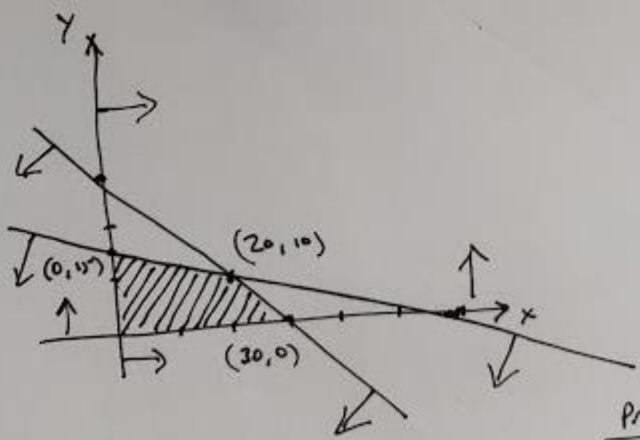
$x + 4(0) = 60$

$x = 60 \quad (60,0)$

Y-Axis $(x=0)$

$(0) + 4y = 60$

$y = 15 \quad (0,15)$



$x + 4y = 60$

$-x - y = -30 \quad (x -1)$

$3y = 30$

$\boxed{y = 10}$

$x + 10 = 30 \quad \therefore \boxed{x = 20}$

Half Planes

$(0) + (0) \le 30$

True $\quad 0 \le 30$

$(0) + 4(0) \le 60$

True $\quad 0 \le 60$

Profit : $x + 3y$

$(0) + 3(15) = 45$

$(30) + 3(0) = 30$

$(20) + 3(10) = 50$

Max profit is €50 for 20 cars, 10 buses

④ Chairs x Tables y Constraints: time (hours), orders

(i) $x \geq 2$, $y \geq 2$, $4x + 8y \leq 40 \Rightarrow x + 2y \leq 10$

$(0) + 2(0) \leq 10$ True!

$x + 2y = 10$

X-AXIS ($y=0$)

$x + 2(0) = 10$

$x = 10$

$(10, 0)$

Y-AXIS ($x = 0$)

$(0) + 2y = 10$

$2y = 10$, $y = 5$

$(0, 5)$

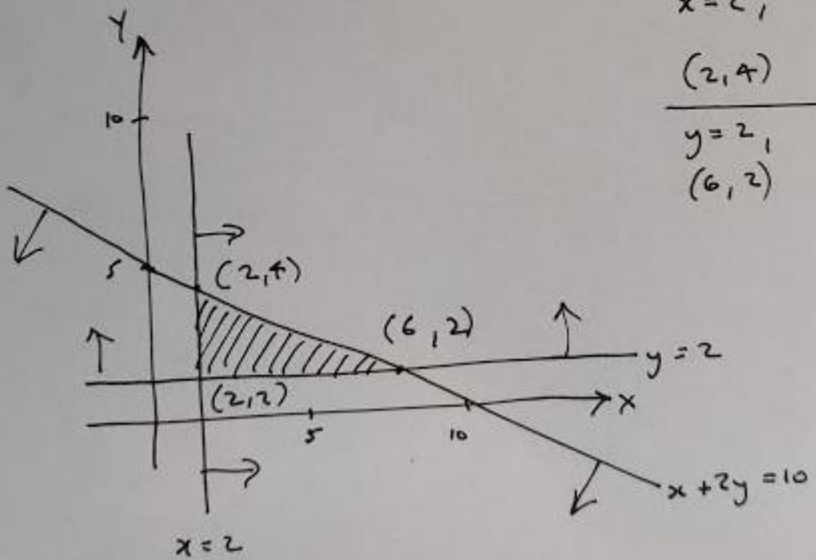
$x = 2$, $(2) + 2y = 10$

$2y = 8$

$(2,4)$ $y = 4$

$y = 2$, $x + 2(2) = 10$

$(6, 2)$ $x = 6$



Profit: $15x + 35y$

$15(2) + 35(2) = 100$

$\boxed{15(2) + 35(4) = 170}$ Max Profit

$15(6) + 35(2) = 160$

(ii) To maximise profit 2 chairs and 4 tables must be produced.

⑤ Bowl A $\quad$ Bowl B $\quad$ Constraints: Hours, Material

$x \geq 0 \qquad y \geq 0$

(Hours) $3x + 2y \leq 42$

(Material) $4x + y \leq 36$

$3x + 2y = 42$

X-AXIS

$3x + 2(0) = 42$

$x = 14 \quad (14,0)$

Y-AXIS

$3(0) + 2y = 42$

$y = 21 \quad (0,21)$

$4x + y = 36$

X-AXIS

$4x + (0) = 36$

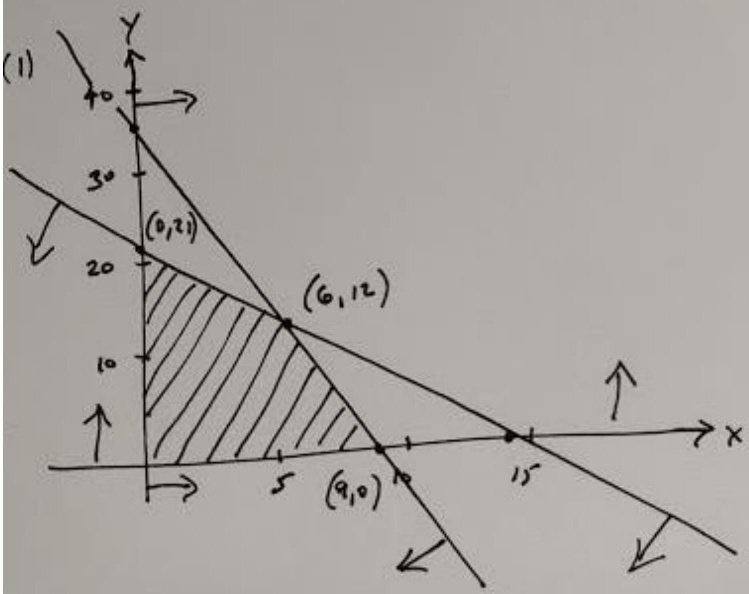
$x = 9 \quad (9,0)$

Y-AXIS

$4(0) + y = 36$

$y = 36 \quad (0,36)$

(I)



$3x + 2y = 42$

$4x + y = 36 \quad (\times -2)$

$3x + \cancel{2y} = 42$

$-8x - \cancel{2y} = -72$

$-5x = -30$

$x = 6$

$4(6) + y = 36$

$y = 36 - 24 = 12$

$(6,12)$

(II) Profit $\quad 14x + 9y$

$14(0) + 9(21) = 189$

$\boxed{14(6) + 9(12) = 192}$ — Max profits

$14(9) + 9(0) = 126$

To maximize profit 6 Bowl A and 12 Bowl B are required.