Using Linear Programming to solve simple mathematical models and mathematical modelling

Today we will do this using straight lines as our equations, and we will solve the problem by drawing these lines (graphing).

This process is called "Linear Programming" and is one of the most powerful mathematical methods used by businesses and companies to solve problems and help them make the best decisions.

"Operations Research" is the profession that applies mathematical methods like this to problems arising in industry, healthcare, finance, etc.

Test Questions 17B

1. Using the same axes and the same scales, indicate the region which simultaneously satisfies the inequalities
$$x \geq 0,\ y \geq 0 \text{ and } x + 2y \leq 8.$$

2. Shade in the set of couples which simultaneously satisfies these inequalities:
$$x \geq 1,\ y \geq 0 \text{ and } 5x + 2y - 10 \leq 0.$$

3. Illustrate the set of points which simultaneously satisfies the inequalities
$$x \geq 0,\ y \geq 1;\ 2x + y \leq 8 \text{ and } 2x + 3y \leq 12.$$

4. Shade in the set of couples (x, y) which simultaneously satisfies these inequalities:
$$y \geq 0;\ 2x - 3y \geq 0 \text{ and } 3x + 5y \leq 15.$$

5. Using the same axes and the same scales indicate the region which simultaneously satisfies the inequalities
$$x \geq 0,\ y \geq 0,\ x + 2y \leq 8 \text{ and } 3x + 2y \leq 12.$$

6. Using the same axes and the same scales plot the inequalities
$$x \geq 0,\ x + 2y \leq 4 \text{ and } x - y - 2 \leq 0.$$
Shade in the region that satisfies all three inequalities simultaneously.

7. Using the same axes and the same scales indicate the region which simultaneously satisfies the inequalities
$$y \geq 0,\ 3x - 2y \geq 0 \text{ and } 3x + 4y - 18 \leq 0.$$
Find the coordinates of the three vertices of this region.

8. Indicate the region which simultaneously satisfies these inequalities:
$$x \geq 1,\ y \geq 0;\ x + y - 6 \leq 0 \text{ and } 3x + y - 12 \leq 0.$$
Now find the coordinates of the four vertices of this region.

2. Shade in the region that simultaneously satisfies the inequalities

$$x \geq 0,\ x - 4y \leq 0 \text{ and } x + y \leq 5.$$

Find (i) the point (x, y) of this region which gives $3x - y$ its maximum value
(ii) the point (x, y) which gives $3x - y$ its minimum value.

3. Using the same axes and the same scales graph the following inequalities:

$$x \geq 0,\ y \geq 0,\ x + 2y - 8 \leq 0 \text{ and } 3x + 2y - 12 \leq 0.$$

Shade in the set of points which simultaneously satisfies these inequalities.
Find a point (x, y) of this set which gives $2x + 3y$ a maximum value.

4. Shade in the region which simultaneously satisfies the following inequalities:

$$x \geq 0,\ y \geq 0,\ 2x + y \leq 8 \text{ and } x + 3y \leq 9.$$

Find the point (x, y) of this region which gives
(i) $2x + 4y$ a maximum value
(ii) $3x - y$ a minimum value.

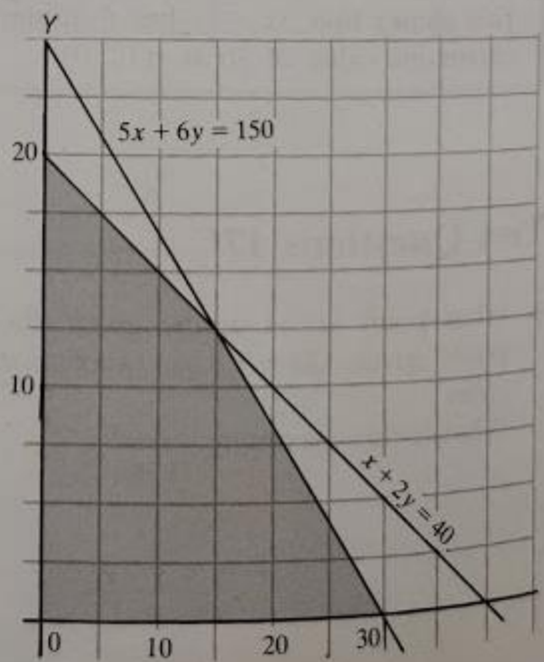
5. Graph the inequalities A, B and C as follows:
A: $y \geq 0$; B: $6x - y \geq 0$ and C: $2x + 3y - 20 \leq 0$.

Shade in the region $A \cap B \cap C$.
Find the point (x, y) of this region which gives
(i) $3x + \frac{1}{2}y$ a maximum value
(ii) $5x - y$ a mimimum value.

6. Graph the following inequalities:
$x \geq 0$; $y \geq 0$; $8x + 12y \leq 96$ and $8x + 6y \leq 72$.

Shade in the region which simultaneously satisfies all four inequalities.
Find the four vertices of this region and hence find the point (x, y) which gives
(i) $4x + 5y$ a maximum value
(ii) $x + 3y$ a minimum value.

7. The point (x, y) lies within or on the boundary of the shaded area shown on the right.



Find (i) the maximum value
(ii) the minimum value
of $3x + y$.

Linear Programming Examples

Example 1

A shopkeeper can buy boxes of chocolates from a wholesaler at IR£10 per box and boxes of sweets at IR£8 per box. When buying his Christmas stock, the shopkeeper wishes to spend not more than IR£800 on these two items. To satisfy orders he has already taken, he needs to buy at least 20 boxes of chocolates and 25 boxes of sweets.

If the shopkeeper's profit is IR£4 on a box of chocolates and IR£2 on a box of sweets, how many of each should be purchased to maximize his profit. Find this profit.

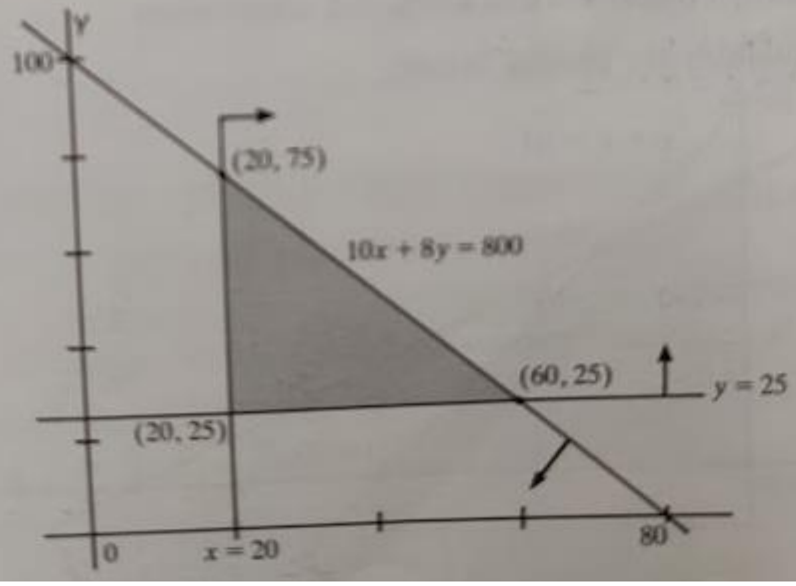
Step 1. Let x be the number of boxes of chocolates and y be the number of boxes of sweets.

Step 2. Write down the inequalities (or constraints) in x and y.

(a) Money constraint: $10x + 8y \leq 800$

(b) To satisfy orders already taken: $x \geq 20$ and $y \geq 25$.

Step 3: Plot these inequalities on a diagram.



The shaded area represents the intersection of the three inequalities.
The three vertices of this region are: (20, 25), (20, 75) and (60, 25).

His profit = IR£$(4x + 2y)$.
We now find the value of $(4x + 2y)$ at each of the vertices.

Vertex	Value of $(4x + 2y)$
(20, 25)	$4(20) + 2(25) = 130$
(20, 75)	$4(20) + 2(75) = 230$
(60, 25)	$4(60) + 2(25) = 290$

So the shopkeeper needs to buy 60 boxes of chocolates and 25 boxes of sweets to maximize his profit. This profit is IR£290, as shown above.

Example 2

A builder is to build not more than 10 houses on an 8000 m^2 site. The houses are of two types—a terraced house which occupies 500 m^2 of ground and a detached house which occupies 1000 m^2 of ground.
Graph the set showing the possible numbers of each type of house that could be built.

The monthly rent he could obtain for the houses is as follows:
IR£400 for the terraced house
IR£600 for the detached house.

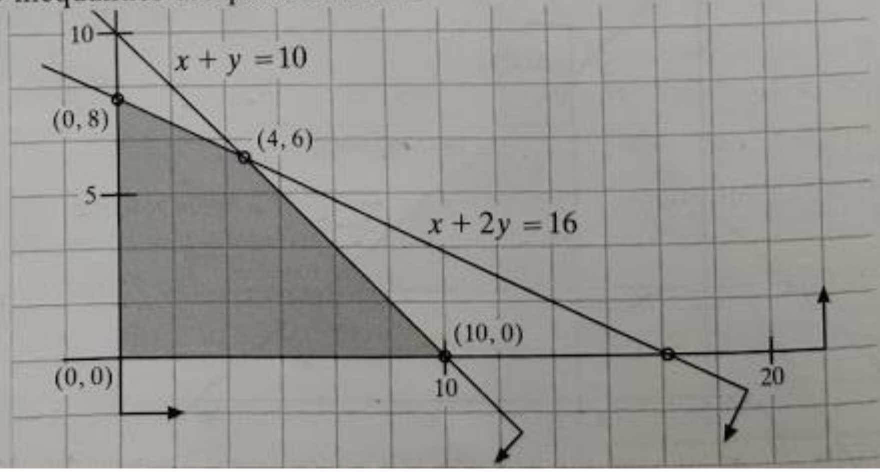
How many of each type should be built to give maximum rent?
Indicate on your graph the region where the rent would exceed IR£3000 per month.

Step 1: Let x be the number of terraced houses and y be the number of detached houses.

Step 2: Ground constraint: $500x + 1000y \leq 8000$.
i.e. $x + 2y \leq 16 \ldots$ (i)
Number constraint: $x + y \leq 10 \ldots$ (ii)

Also $x \geq 0$ and $y \geq 0$ (understood constraints)

These inequalities are plotted below:



The shaded region shows the possible numbers of each type of house that could be built.

The four vertices of this region are (0, 0), (10, 0), (4, 6) and (0, 8).

The rent is IR£$(400x + 600y)$.
We now find the value of $400x + 600y$ at each vertex.

Vertex	Value of $400x + 600y$
(0, 0)	0
(10, 0)	4000
(4, 6)	5200
(0, 8)	4800

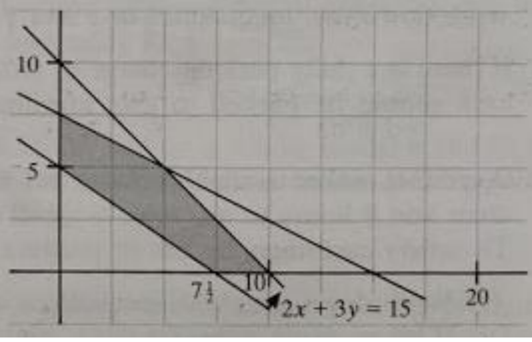
So 4 terraced houses (x) and 6 detached houses (y) should be built to maximize the rent. This maximum is IR£5200.

If the rent is to exceed IR£3000 per month, this gives the inequality

$$400x + 600y > 3000$$
$$\text{i.e. } 2x + 3y > 15$$

This inequality is graphed in the diagram below.

The shaded portion of the diagram on the right represents the region where the rent is greater than IR£3000.



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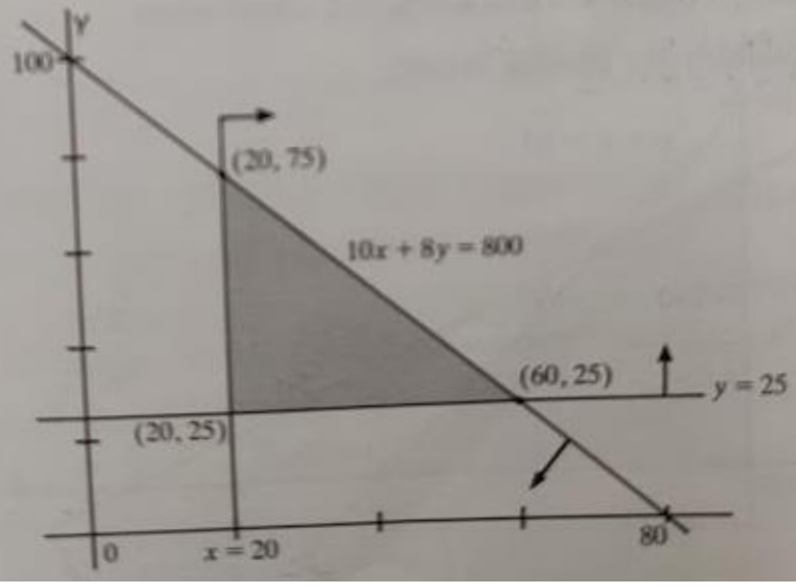

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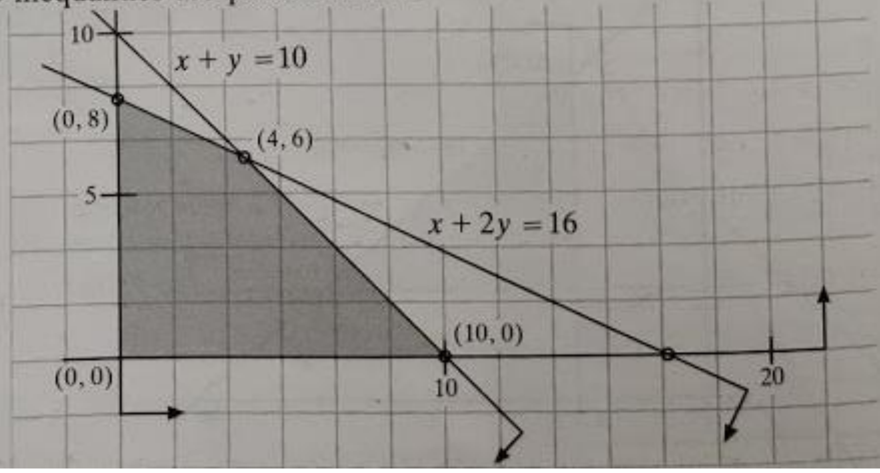

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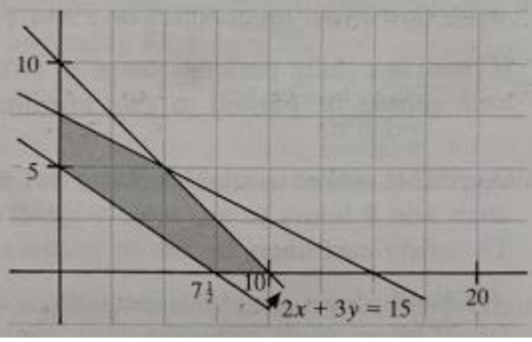


Linear Programming Exercises - Several

1. A manager plans to buy two types of machine, type A and type B, for his factory. He has 88 m^2 of space available for these machines. Type A needs 6 m^2 of floor space while type B needs 4 m^2 of floor space.
   Type A costs IR£80 each and type B costs IR£400 each and the manager has up to IR£3600 to spend.
   Assuming $x \geq 0$ and $y \geq 0$, write down two other inequalities in x and y.
   Hence find the greatest number of machines he can buy.

2. A factory produces fridges and freezers. In any given week at most 200 of these products can be produced. Regular customers must be supplied with at least 20 fridges and 50 freezers per week.
   Write down three inequalities to illustrate these constraints and then graph the inequalities.

   If the profit on a fridge is IR£40 and the profit on a freezer is IR£50, how many of each should be produced to maximize profits.

3. A hotel carpark is 360 m^2. The parking area required for a car is 6 m^2 and for a bus 24 m^2. Not more than 30 vehicles could be accommodated at any given time. If x is the number of cars and y is the number of buses to be accommodated, write down two inequalities in x and y to illustrate these constraints.

   If there is a daily parking charge of IR£1 per car and IR£3 per bus, how many of each should be parked to give maximum daily income, and state this income.

4. A cabinet maker assembles chairs and tables. It takes him 4 hours to assemble a chair and 8 hours to assemble a table, and he works 40 hours per week.
   To satisfy customers he has to produce 2 chairs and 2 tables each week.

   (i) Write down the three inequalities and illustrate them on a graph.

   (ii) If his profit on a chair is IR£15 and his profit on a table is IR£35, how many of each should he assemble each week so as to maximize his profits?

5. A crystal glass factory produces two types of bowl, A and B. Each bowl A takes 3 hours on machine and 4 units of raw materials.
   Each bowl B takes 2 hours on machine and 1 unit of raw materials.
   In any week the factory has only 42 machine hours and 36 units of raw materials available.

   If x of bowl A and y of bowl B are produced each week, write down two inequalities in x and y (other than $x \geq 0$ and $y \geq 0$).

   (i) Illustrate these inequalities on a graph.

   (ii) If the profit on each bowl A is IR£14 and on bowl B IR£9, calculate how many of each should be produced each week to maximize profits.

① A B

$x \geq 0$ $y \geq 0$

Constraints – Space

– Money

$6x + 4y \leq 88 \longrightarrow 3x + 2y = 44$

$80x + 400y \leq 3600 \longrightarrow x + 5y = 45$

$3x + 2y = 44$

X-AXIS ($y=0$)

$3x + 2(0) = 44$

$3x = 44$

$x = 14.\dot{6}$

$(14.6, 0)$

Y-AXIS ($x=0$)

$3(0) + 2y = 44$

$2y = 44$

$y = 22$

$(0, 22)$

$x + 5y = 45$

X-AXIS ($y=0$)

$x + 5(0) = 45$

$x = 45$

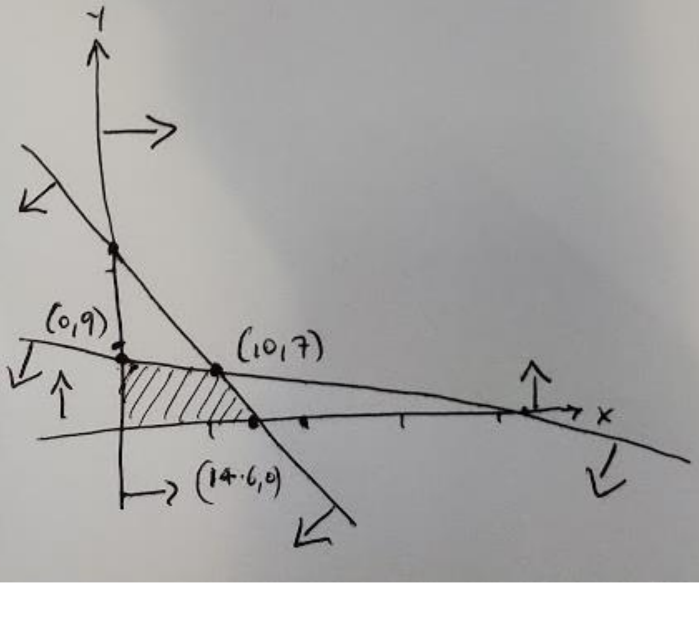
$(45, 0)$

Y-AXIS ($x=0$)

$(0) + 5y = 45$

$y = 9$

$(0, 9)$



$3x + 2y = 44$

$x + 5y = 45$ $(\times -3)$

$3x + 2y = 44$

$-3x - 15y = -135$

$-13y = -91$

$y = \frac{-91}{-13} = 7$

$x + 5(7) = 45$

$\therefore x = 10$ $(10, 7)$

10 + 7 = 17 machines

② Fridge Freezers

$x \geq 20$ $y \geq 50$

$x + y \leq 200$

$(0) + (0) \leq 200$ (True)

$x + y = 200$

x-Axis $(y = 0)$

$x + (0) = 200$

$x = 200$

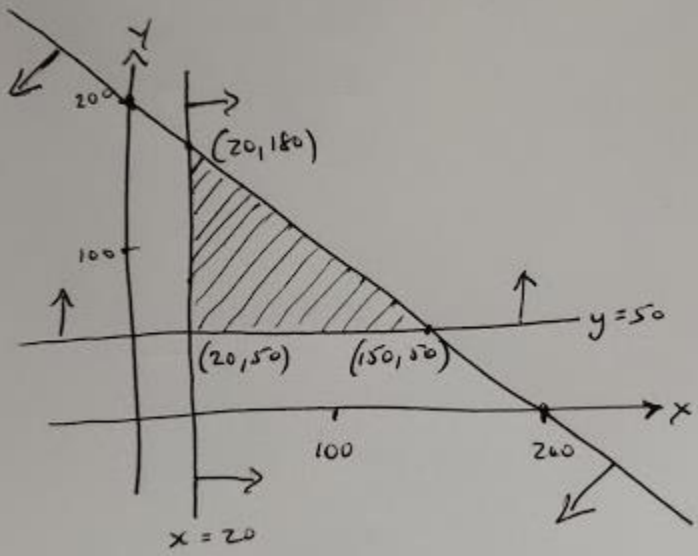
$(200, 0)$

Y-Axis $(x = 0)$

$(0) + y = 200$

$y = 200$

$(0, 200)$



$x = 20$ $(20) + y = 200$

$y = 180$

$(20, 180)$

$Y = 50$ $x + (50) = 200$

$x = 150$

$(150, 50)$

Profit $40x + 50y$

$40(20) + 50(50) = 3300$

$40(20) + 50(180) = 9800$

$40(150) + 50(50) = 8500$

Max Profit = £9,800

③ Car Bus

x y

Constraints — Area, number of vehicles

(i) $6x + 24y \leq 360$ $\div 6$ $\Rightarrow$ $\boxed{x + 4y \leq 60}$

(ii) $\boxed{x + y \leq 30}$ $\boxed{x \geq 0}$ $\boxed{y \geq 0}$

Maximise profit $\Rightarrow$ $\boxed{x + 3y}$

$x + y = 30$

X-Axis $(y = 0)$

$x + (0) = 30$

$x = 30$ $(30, 0)$

Y-Axis $(x = 0)$

$(0) + y = 30$

$y = 30$ $(0, 30)$

$x + 4y = 60$

X-Axis $(y = 0)$

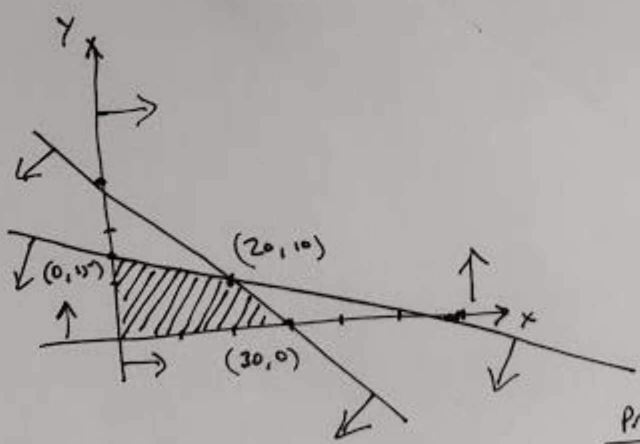
$x + 4(0) = 60$

$x = 60$ $(60, 0)$

Y-Axis $(x = 0)$

$(0) + 4y = 60$

$y = 15$ $(0, 15)$



$x + 4y = 60$

$-x - y = -30$ $(\times -1)$

$3y = 30$

$\boxed{y = 10}$

$x + 10 = 30$ $\therefore \boxed{x = 20}$

Half Planes

$(0) + (0) \leq 30$

True $0 \leq 30$

$(0) + 4(0) \leq 60$

True $0 \leq 60$

Profit : $x + 3y$

$(0) + 3(15) = 45$

$(30) + 3(0) = 30$

$(20) + 3(10) = 50$

Max profit is €50 for 20 cars, 10 buses

④ Chairs Tables Constraints: time (hours), orders

x y

(i) $x \geq 2$, $y \geq 2$, $4x + 8y \leq 40 \Rightarrow x + 2y \leq 10$

$(0) + 2(0) \leq 10$ True!

$x + 2y = 10$

X-AXIS $(y=0)$

$x + 2(0) = 10$

$x = 10$

$(10, 0)$

Y-AXIS $(x=0)$

$(0) + 2y = 10$

$2y = 10$, $y = 5$

$(0, 5)$

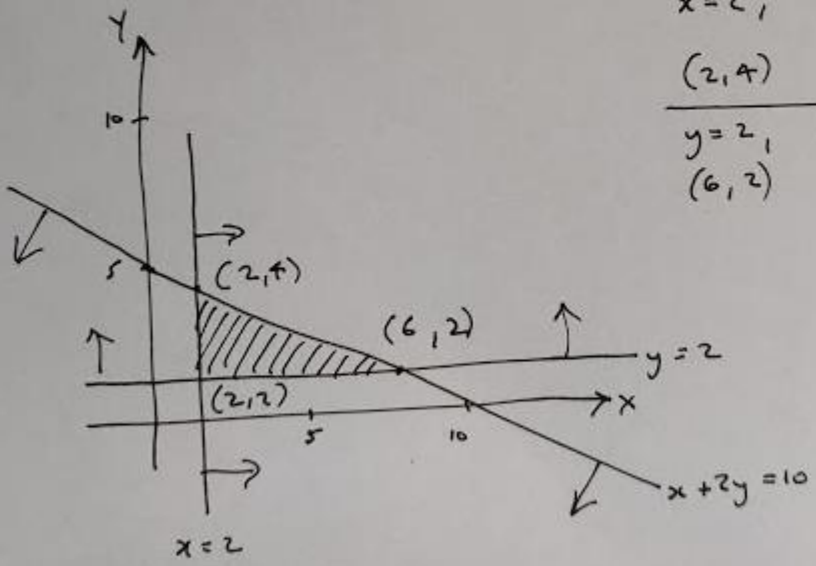
$x = 2$, $(2) + 2y = 10$

$2y = 8$

$(2, 4)$ $y = 4$

$y = 2$, $x + 2(2) = 10$

$(6, 2)$ $x = 6$



Profit: $15x + 35y$

$15(2) + 35(2) = 100$

$15(2) + 35(4) = 170$ Max Profit

$15(6) + 35(2) = 160$

(ii) To maximise profit 2 chairs and 4 tables must be produced.

⑤ Bowl A Bowl B Constraints: Hours, Material

$x \geq 0$ $y \geq 0$

(Hours) $3x + 2y \leq 42$

(Material) $4x + y \leq 36$

$3x + 2y = 42$

X-AXIS

$3x + 2(0) = 42$

$x = 14$ (14, 0)

Y-AXIS

$3(0) + 2y = 42$

$y = 21$ (0, 21)

$4x + y = 36$

X-AXIS

$4x + (0) = 36$

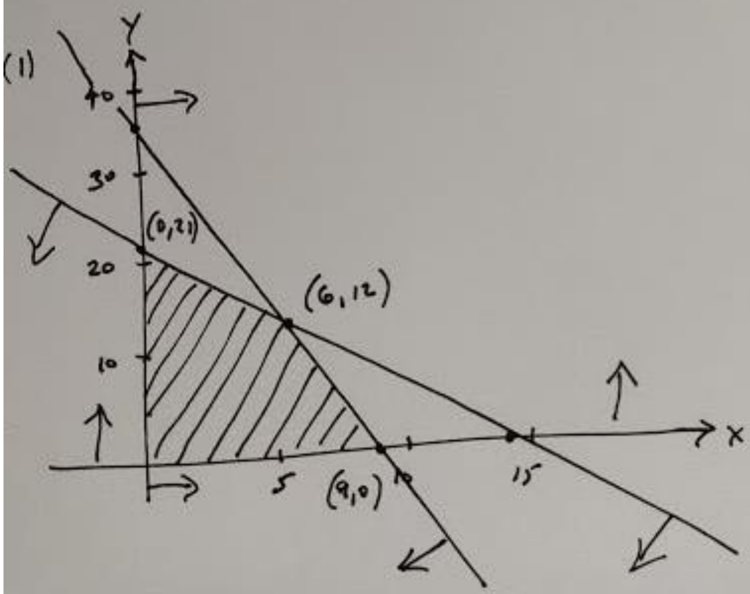
$x = 9$ (9, 0)

Y-AXIS

$4(0) + y = 36$

$y = 36$ (0, 36)

(I)



$3x + 2y = 42$

$4x + y = 36$ (x −2)

$3x + 2y = 42$

$-8x - 2y = -72$

$-5x = -30$

$x = 6$

$4(6) + y = 36$

$y = 36 - 24 = 12$

(6, 12)

(II) Profit $14x + 9y$

$14(0) + 9(21) = 189$

$14(6) + 9(12) = 192$ — Max profits

$14(9) + 9(0) = 126$

To maximise profit 6 Bowl A and 12 Bowl B are required.