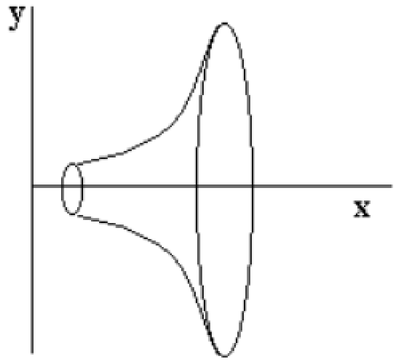
Integration – Volume

To find volume generated by rotating a shape about the x-axis $\quad Vol = \int_a^b \pi y^2$

Example 1

The graph of $y = x^2$ between $x = 1$ and $x = 3$ is rotated completely about the x-axis. Find the volume generated.

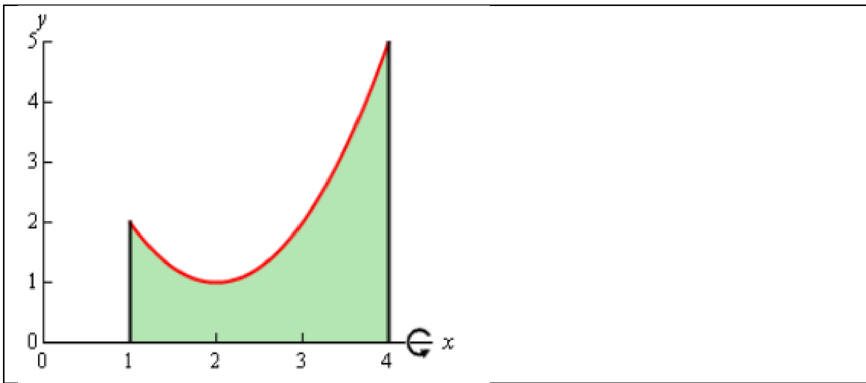


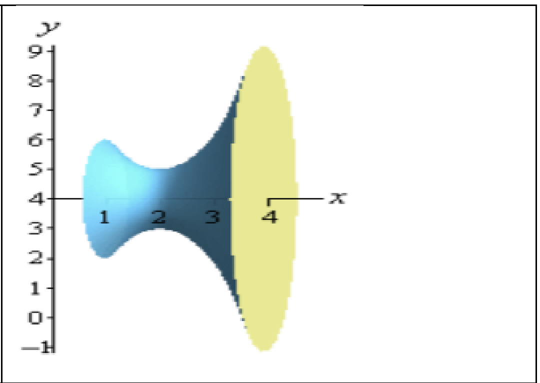
$\int_1^3 \pi x^4 dx \quad = \pi \frac{x^5}{5} + c \ \ between\ 1\ and\ 3 =$

$= \pi(\frac{(3)^5}{5} - \frac{(1)^5}{5}) = \frac{243\pi}{5} - \frac{\pi}{5} = \frac{242\pi}{5} = 48.4\pi\ units^3$

Example 2

Determine the volume of the solid generated by rotating the region bounded by $f(x) = x^2 - 4x + 5$, x = 1, x = 4 and the x-axis about the x-axis.





$\int_1^4 \pi (x^2 - 4x + 5)^2 dx \quad = \pi(x^4 - 8x^3 + 26x^2 - 40x + 25 + c)\ \ between\ 1\ and\ 4 =$

$= \pi((\frac{(4)^5}{5} - \frac{8(4)^4}{4} + \frac{26(4)^3}{3} - \frac{40(4)^2}{2} + 25(4) + c) - (\frac{(1)^5}{5} - \frac{8(1)^4}{4} + \frac{26(1)^3}{3} - \frac{40(1)^2}{2} + 25(1) + c))$

$= \pi((\frac{1024}{5} - \frac{2048}{4} + \frac{1664}{3} - \frac{460}{2} + 100 + c - \frac{1}{5} + \frac{8}{4} - \frac{26}{3} + \frac{40}{2} - 25 - c))$

$= \pi((\frac{1024}{5} - 512 + \frac{1664}{3} - 130 + 100 + c - \frac{1}{5} + 2 - \frac{26}{3} + 20 - 25 - c))$

$= \frac{78\pi}{5}\ units^3$

Integration – Area & Volume Exercises

By using a definite integral find the area of the region bounded by the given curves:

Questions		Answers	
		4	
a)	$y=-x^2+2x+8,\ y=0$	**a)**	36
b)	$y=16-x^2,\ y=x^2-16$	**b)**	$18\sqrt{2}\doteq 25.456$
c)	$y=x^2-4x+6,\ y=-2x^2+8x-3$	**c)**	$\frac{1}{6}$
d)	$y=x^2+6x+8,\ y=-x^2-10x-16$	**d)**	$\frac{1}{3}$
e)	$y=-9-x^2,\ 5x+y+9=0$	**e)**	$3-e\doteq 0.282$
f)	$y=x^2,\ y=2x^2,\ y=1$	**f)**	1
g)	$y=2x^3,\ y=\frac{x}{2}$	**g)**	$\frac{9}{2}$
h)	$y=x,\ y=3\sqrt{x}$	**h)**	$\pi\doteq 3.142$
i)	$y=x^2+1,\ y=0,\ x=-1,\ x=2$	**i)**	$\frac{4}{3}$
j)	$y=-x^2+2x-3,\ y=0,\ x=0,\ x=3$	**j)**	$\frac{16}{3}$
k)	$y=\sin x,\ y=0,\ x=0,\ x=2\pi$	**k)**	$\frac{\pi}{2}-\frac{1}{3}\doteq 1.237$

5	By using a definite integral find the volume of the solid obtained by rotating the region bounded by the given curves around the x-axis :

a) $y = x,\ y = \dfrac{1}{x},\ y = 0,\ x = 2$

b) $y = x^2,\ y = x$

c) $y = x^3 + 3,\ y = 0,\ x = -1,\ x = 1$

d) $y = \dfrac{4}{x},\ y = 0,\ x = 1,\ x = 4$

e) $y = -x^2 + 1,\ y = -2x^2 + 2$

f) $y = \dfrac{1}{1+x^2},\ x = -1,\ x = 1$

g) $y = x^2 - 6x + 9,\ y = x^2 - 4x + 7,\ x = 3$

h) $y = \sqrt{2x-3},\ y = \sqrt{4x-7},\ y = 0$

i) $y = 2^x,\ 3x - 4y + 5 = 0$

k) $y = \sin x,\ y = 0,\ x \in \langle 0;\pi \rangle$

l) $y = \sin 2x,\ y = 0,\ x \in \left\langle 0;\dfrac{\pi}{2} \right\rangle$

m) $y = \sin^2 x,\ y = 3\sin x,\ x \in \langle 0;\pi \rangle$

n) $y = \sqrt{x}\, e^{-x},\ y = 0,\ x = 1$

o) $y = \dfrac{\ln x}{x},\ y = 0,\ x = 1,\ x = e$

p) $y = e^{2x},\ y = e^{-2x},\ y = e^2$

r) $x^2 + y^2 = 4,\ x \in \langle -2;2 \rangle$

s) $y^2 = 5x,\ x = 8$

t) $y = x^2,\ x = y^3$

Answers

5

a) $\dfrac{5}{6}\pi \doteq 2.618$

b) $\dfrac{\pi}{30} \doteq 0.105$

c) $\dfrac{128}{7}\pi \doteq 18.286$

d) $12\pi \doteq 37.699$

e) $\dfrac{16}{5}\pi \doteq 10.053$

f) $\dfrac{\pi}{4}(\pi + 2) \doteq 4.038$

g) $16\pi \doteq 50.265$

h) $\dfrac{\pi}{8} \doteq 0.393$

i) $\dfrac{\pi}{2}\left(7 - \dfrac{15}{4\ln 2}\right) \doteq 2.497$

k) $\dfrac{\pi^2}{2} \doteq 4.935$

l) $\dfrac{\pi^2}{4} \doteq 2.467$

m) $\dfrac{33}{8}\pi^2 \doteq 40.712$

n) $\dfrac{\pi}{4} - \dfrac{3\pi}{4e^2} \doteq 0.467$

o) $2\pi - \dfrac{5\pi}{e} \doteq 0.505$

p) $\dfrac{\pi}{2}(3e^4 + 1) \doteq 258.859$

r) $16\pi \doteq 50.265$

s) $160\pi \doteq 502.655$

t) $\dfrac{2}{5}\pi \doteq 1.257$

6 By using a definite integral find the volume of the solid obtained by rotating the region bounded by the given curves around the y-axis :

a) $y = 4 - x^2, \; x = 0, \; x = 2$

b) $y = x^2 - x - 6, \; x = 3, \; x = 5$

c) $y = \dfrac{3}{x^4}, \; x = 1, \; x = 4$

d) $y = e^{-x}, \; y = 0, \; x = 0, \; x = 1$

e) $y = \sin x, \; y = \dfrac{1}{2}, \; x = 0$

f) $y = \ln x, \; y = 0, \; y = 1, \; x = 0$

g) $y = \dfrac{5}{x^3}, \; x = 1, \; x = 4$

h) $y = e^{-2x}, \; y = 0, \; x = 0, \; x = 2$

i) $y = 4x^3, \; y = x^2, \; x = 1, \; x = 5$

j) $y = \dfrac{5}{x^2}, \; x = 1, \; x = \dfrac{9}{2}$

k) $y = x^2 - 6x + 9, \; x = 3, \; x = 5$

l) $y = x^2 - 3x, \; x = 3, \; x = 5$

m) $y = 4x - x^2, \; x = 0, \; x = 4$

n) $y = \dfrac{3}{x^2 + 1}, \; x = 0, \; x = 3$

o) $y = \dfrac{x^2}{2}, \; y = \dfrac{|x|}{2}$

p) $y = x^2, \; x = y^2$

r) $4y = x^2, \; 4x = y^2$

s) $y^2 = x^3, \; y = 0, \; x = 1$

t) $y^2 = 4 - x, \; x = 0$

u) $y^2 = x^2 - x^3$

Answers

6

a) $8\pi \doteq 25.133$

b) $\dfrac{332}{3}\pi \doteq 347.670$

c) $\dfrac{45}{16}\pi \doteq 8.836$

d) $2\pi\left(1 - \dfrac{2}{e}\right) \doteq 1.660$

e) $\dfrac{1}{72}\pi^3 + \dfrac{\sqrt{3}}{6}\pi^2 - \pi \doteq 0.138$

f) $\dfrac{\pi}{2}\left(e^2 - 1\right) \doteq 10.036$

g) $\dfrac{15}{2}\pi \doteq 23.562$

h) $\dfrac{\pi}{2}\left(1 - \dfrac{5}{e^4}\right) \doteq 1.427$

i) $\dfrac{23432}{5}\pi \doteq 14{,}722.760$

k) $24\pi \doteq 75.398$

l) $76\pi \doteq 238.761$

m) $\dfrac{128}{3}\pi \doteq 134.041$

n) $3\pi \ln 10 \doteq 21.701$

o) $\dfrac{\pi}{12} \doteq 0.262$

p) $\dfrac{3}{10}\pi \doteq 0.942$

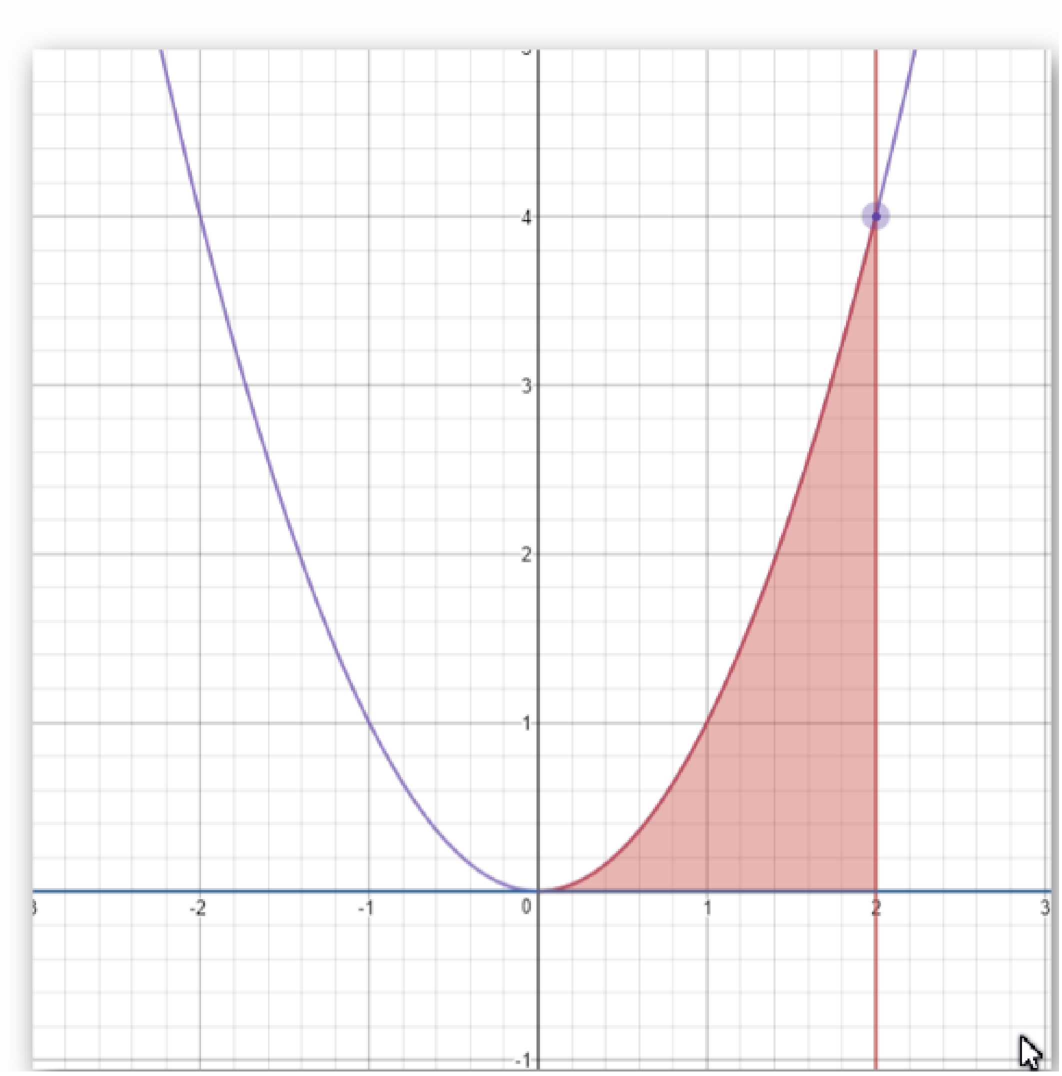
r) $\dfrac{96}{5}\pi \doteq 60.319$

s) $\dfrac{4}{7}\pi \doteq 1.795$

t) $\dfrac{512}{15}\pi \doteq 107.233$

How do I find the volume of the solid generated by revolving the region bounded by $y=x^2$, $y=0$, and $x=2$ about the x-axis? The y-axis?

Explanation:
the rose region is revolving about the x-axis and y-axis



1)when the shaded region revolving a bout x-axis

$$Volume = \pi \int_a^b y^2 \cdot dx$$

$$Volume = \pi \int_0^2 y^2 \cdot dx = \pi \int_0^2 x^4 \cdot dx = \pi \left[\frac{1}{5} \cdot x^5\right]_0^2$$

$$= \pi\left[\left(\frac{32}{5}\right) - 0\right] = \left(\frac{32}{5}\right)\pi(unite)^3$$

2)when the shaded region revolving about the y-axis

$$Volume = \pi \int_d^c \left[\left(x^2\right)_2 - \left(x^2\right)_1\right] \cdot dy$$

$$Volume = \pi \int_0^4 \left[\left(2^2\right)_2 - \left(\sqrt{y}^2\right)_1\right] \cdot dy$$

$$= \pi \int_0^4 [4 - y] \cdot dy = \pi\left[4y - \frac{1}{2}y^2\right]_0^4$$

$$= \pi[16 - 8] = 8\pi(unite)^3$$

Q4 a

$y = -x^2 + 2x + 8$

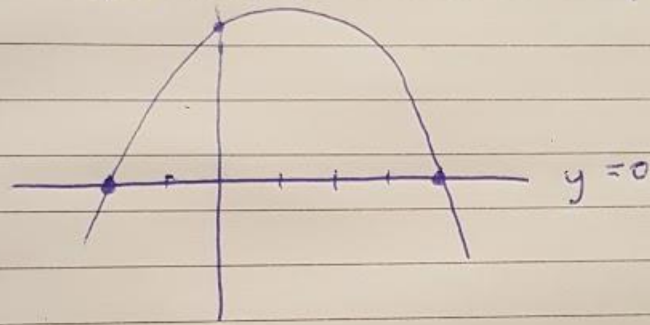
$0 = (-x + 4)(x + 2)$

$0 = -x + 4$, $x + 2 = 0$

$x = 4$ $\quad$ $x = -2$

$(4,0)$ $\quad$ $(-2,0)$

$f(0) = -(0)^2 + 2(0) + 8 = 8$ $\quad$ $(0,8)$



$$\int_{-2}^{4} (-x^2 + 2x + 8)\,dx = \left[-\frac{x^3}{3} + \frac{2x^2}{2} + 8x + c \right]_{-2}^{4}$$

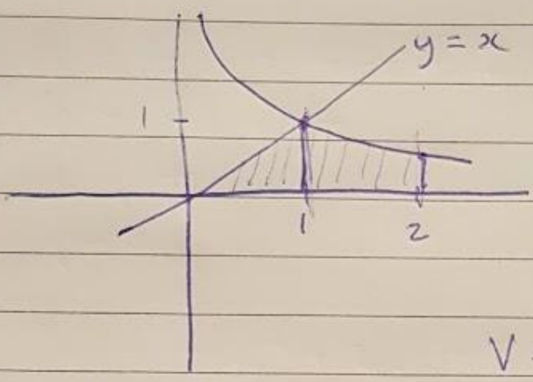
$$\left[-\frac{(4)^3}{3} + \frac{2(4)^2}{2} + 8(4) + c \right] - \left[\frac{-(-2)^3}{3} + (-2)^2 + 8(-2) + c \right]$$

$$-\frac{64}{3} + 16 + 32 + \cancel{c} - \frac{8}{3} - 4 + 16 - \cancel{c}$$

$$60 - \frac{72}{3} = 60 - 24 = 36 \text{ units}^2$$

Q5 a

5(a) $y = x$, $y = \frac{1}{x}$ $y = 0$, $x = 2$



$$V = \pi \int y^2 \, dx$$

$$\pi \int_0^1 x^2 \, dx = \left| \frac{x^3}{3} + c \right|_0^1 =$$

$$\pi\left(\frac{(1)^3}{3} + c - \frac{(0)^3}{3} - c\right) = \pi \frac{1}{3}$$

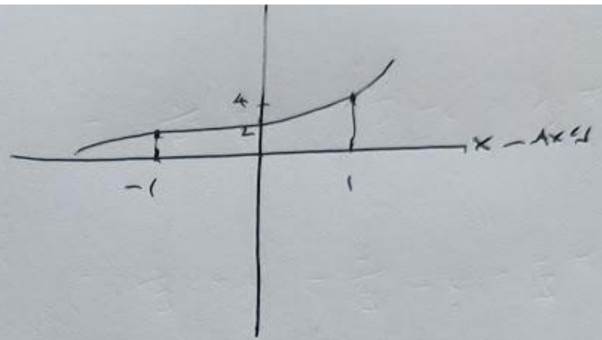
$$\pi \int_1^2 \frac{1}{x^2} \, dx = x^{-2} \Rightarrow \frac{x^{-1}}{-1} + c = \pi \left| \frac{1}{-x} + c \right|_1^2$$

$$\pi \left(\frac{1}{-2} + \cancel{c} - \frac{1}{-1} \mp \cancel{c} \right) = +\frac{1}{2}\pi$$

$$\frac{1}{3}\pi + \frac{1}{2}\pi = \frac{2}{6}\pi + \frac{3}{6}\pi = \frac{5}{6}\pi$$

Q5c

$$y = x^3 + 3$$
$$y^2 = (x^3+3)(x^3+3)$$
$$y^2 = x^6 + 3x^3 + 3x^3 + 9$$
$$y^2 = x^6 + 6x^3 + 9$$



$$\pi\int_{-1}^{1} x^6 + 6x^3 + 9 = \pi\left(\frac{x^7}{7} + \frac{6x^4}{4} + 9x\right)_{-1}^{1}$$

$$\pi\left(\left(\frac{1}{7} + \frac{6}{4} + 9\right) - \left(\frac{-1}{7} + \frac{6}{4} - 9\right)\right)$$

$$\frac{1}{7} + \cancel{\frac{6}{4}} + 9 + \frac{1}{7} - \cancel{\frac{6}{4}} + 9$$

$$18\tfrac{2}{7}\pi$$

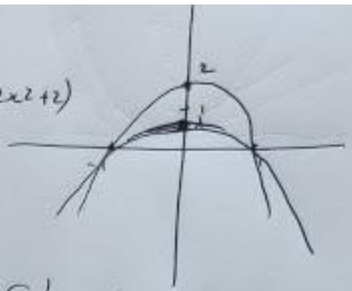
$$\frac{128}{7}\pi$$

Q5e

$y = -x^2+1$, $y = -2x^2+2$

$y^2 = (-x^2+1)(-x^2+1)$, $y^2 = (-2x^2+2)(-2x2+2)$

$y^2 = x^4 + 2x^2 + 1$



$$\pi\int_{-1}^{1} 4x^4 - 8x^2 + 4 \quad - \pi\int_{-1}^{1} (x^4 - 2x^2 + 1)$$

① ②

$$\pi\left(\frac{4x^5}{5} - \frac{8x^3}{3} + 4x\right) - \pi\left(\frac{x^5}{5} - \frac{2x^3}{3} + x\right)$$

$$\pi\left(\left(\frac{4(1)^5}{5} - \frac{8(1)^3}{3} + 4(1)\right) - \left(\frac{4(-1)^5}{5} - \frac{8(-1)^3}{3} + 4(-1)\right)\right)$$

$$\pi\left(\frac{4}{5} - \frac{8}{3} + 4 + \frac{4}{5} - \frac{8}{3} + 4\right) =$$

$$8 + \frac{8}{5} - \frac{16}{3} \Rightarrow \frac{120}{15} + \frac{24}{15} - \frac{80}{15} = \frac{64}{15}\pi$$

$$\pi\left(\left(\frac{(1)^5}{5} - \frac{2(1)^3}{3} + (1)\right) - \left(\frac{(-1)^5}{5} - \frac{2(-1)^3}{3} + (-1)\right)\right)$$

$$\pi\left(\frac{1}{5} - \frac{2}{3} + 1 + \frac{1}{5} - \frac{2}{3} + 1\right)$$

$$= 2 + \frac{2}{5} - \frac{4}{3} = \frac{30}{15} + \frac{6}{15} - \frac{20}{15} = \frac{16}{15}\pi$$

$$\frac{64}{15}\pi - \frac{16}{15}\pi = \frac{48}{15}\pi = \frac{16}{5}\pi$$

Q6 a

6(a) $y = 4 - x^2$, $x = 0$, $x = 2$

About y axis $\quad \pi \int x^2 \, dy$.

$y = 4 - x^2$

$y - 4 = -x^2$

$\boxed{x^2 = -y + 4}$

$x^2 = 4 - y$

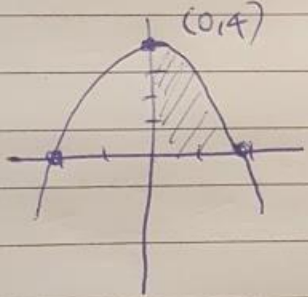
$0 = 4 - x^2$

$x^2 = 4$

$x = \sqrt{4}$

$x = \pm 2$

Let $x = 0$, $y = 4 - (0)^2 = y = 4$, $(0, 4)$



$\pi \int (4 - y) \, dy = \pi \left[4y - \frac{y^2}{2} + c \right]_0^4$

$\pi \left[\left(4(4) - \frac{(4)^2}{2} + c \right) - \left(4(0) - \frac{(0)^2}{2} + c \right) \right]$

$\pi \left[16 - 8 + \cancel{c} - 0 - 0 - \cancel{c} \right]$

8π