4. As a car passes a point p, its driver applies the brakes. The car's distance, s, from p at any subsequent time, t, is given by

$$s(t) = 20t - 2t^2$$

where s is measured in metres and t in seconds.

Find

(i) the car's distance from p at $t = 4$.

(ii) the car's speed at $t = 4$.

(iii) the time when the car comes to rest.

(iv) the car's distance from p when it stops.

(v) the constant deceleration of the car.

5. As soon as an aeroplane touches down, it applies brakes. The distance, s, which it has travelled along the runway at time t seconds after touchdown is given by

$$s(t) = 200t - 4t^2 \quad \text{metres.}$$

Find

(i) the speed of the aeroplane at $t = 3$.

(ii) the speed of the aeroplane at $t = 4$.

(iii) the constant deceleration of the aeroplane.

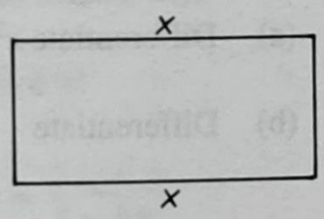
(iv) the time taken in coming to rest.

(v) the distance travelled by the plane before coming to rest.

6. A piece of wire, 40 cm long, is bent to form a rectangle.
If the length of the rectangle is x, show that its area, A, is given by

$$A = 20x - x^2.$$

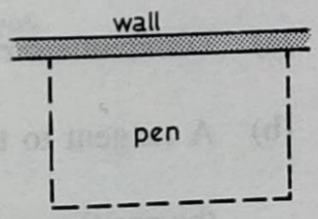
Hence find the maximum possible area.



7. A straight wall runs along one side of a farm. The farmer has 60 m of fencing to complete the other 3 sides of a small rectangular pen.
If x = the width of the pen, show that the area, A, is given by

$$A(x) = 60x - 2x^2.$$

Hence find the maximum possible area of the pen.



Solutions

Q4 $s(t) = 20t - 2t^2$

(i) $t = 4$, $s(4) = 20(4) - 2(4)^2 = 80 - 32 = \boxed{48m}$

(ii) $\frac{ds}{dt} = 20 - 4t$,

$20 - 4(4) = 20 - 16 = \boxed{4 m/s}$

(iii) stopped, speed $= 0$

$20 - 4t = 0$

$20 = 4t$, $\boxed{t = 5s}$

(iv) $s(5) = 20(5) - 2(5)^2$

$100 - 50$ $\boxed{50m}$

(v) $\frac{d^2s}{dt^2} = -4$ $\boxed{4m/s^2}$ Deceleration

Q5 $s(t) = 200t - 4t^2$ Speed $= \frac{ds}{dt} = 200 - 8t$

(i) $200 - 8(3) = \boxed{176 m}$

(ii) $200 - 8(4) = \boxed{168m}$

(iii) $\frac{d^2s}{dt^2} = -8m/s^2$ Deceleration of $8 m/s^2$

(iv) $200 - 8t = 0$

$200 = 8t$

$25 = t$ $t = \boxed{25s}$

(v) $s(25) = 200(25) - 4(25)^2$

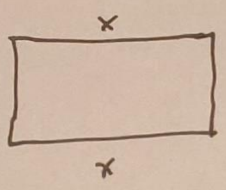
$5000 - 4(625) =$

$5000 - 2500$

$= \boxed{2500m}$

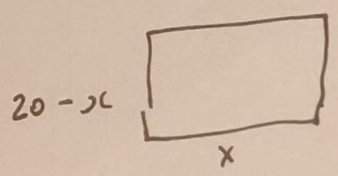
Solutions

Q6



$40 - 2x$ = Both sides

$\therefore$ one side is $\frac{40-2x}{2} = 20 - x$



$\therefore A = x(20-x)$

$A = 20x - x^2$

$\frac{dA}{dx} = 20 - 2x$

$20 - 2x = 0$

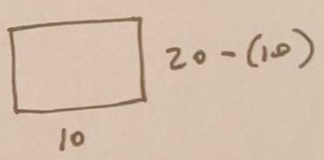
$20 = 2x$

$10\text{cm} = x$

$\frac{d^2A}{dx^2} = -2$

$\therefore$ Max

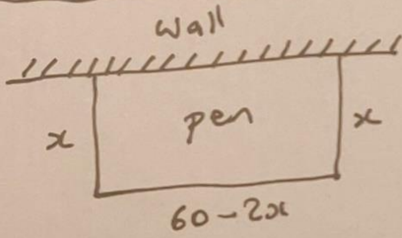
(+ = minimum)



Area $= 10 \times 10$

$= \boxed{100\text{ cm}^2}$

Q7



$60 - 2x$ (length)

Area $= x(60-2x)$

$= 60x - 2x^2$

$\frac{dA}{dx} = 60 - 4x$

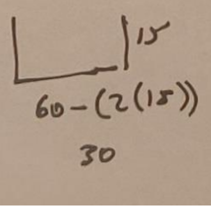
$\frac{d^2A}{dx^2} = -4$

($\therefore$ a maximum)

$60 - 4x = 0$

$60 = 4x$

$15 = x$



Area $= 15 \times 30$

$= \boxed{450\text{ m}^2}$

Applications of Differentiation – Turning Points

Find the local maximum and minimum turning points for the following function

$$f(x) = x^3 + 6x^2 + 9x + 2$$

(c) Find the slope and hence the equation of the tangent to the curve $y = (3x-2)(1-x^2)$ at the point $(0, -2)$.

(d) Find the coordinates of the local maximum of the curve $y = 4x^3 - 3x + 4$.

4. (a) If $y = \dfrac{x^2-4}{2x-3}$, find the value of $\dfrac{dy}{dx}$ when $x = 0$.

(b) If $y = (2x-1)(x^2-2)$, find the values of x for which $\dfrac{dy}{dx} = 0$.

(c) Find the slope of the tangent to the curve $y = (2x^2 - 4x)^3$ at $x = 1$.

(d) The distance s metres travelled by a body in t seconds is given by the formula

$$s = 3 - 6t + 2t^3.$$

(i) Find the speed of the body after 2 seconds.
(ii) Show that the body is stopped when $t = 1$.
(iii) Find the acceleration after 3 seconds.

5. (a) Differentiate $2 - 3x - x^2$ with respect to x from first principles.

(b) Find the point on the curve $y = 4x - x^2$ at which the slope of the tangent to the curve is 1.

(c) Find the coordinates of the local maximum of the function

$$y = x^3 - 9x^2 + 24x - 20.$$

6. (a) If $y = (2x-3)^3$, find the value of $\dfrac{dy}{dx}$ when $x = 1$.

(b) Show that $(1, -5)$ is a point of the curve $y = 2x^2 - 5x - 2$, and find the equation of the tangent to the curve at this point.

(c) Show that the curve $y = x^3 - 3x + 5$ has a local minimum at the point $(1, 3)$. Find the coordinates of the local maximum.

7. (a) Differentiate $3x^2 - x + 4$ with respect to x from first principles.

(b) If $y = (1-3x^2)^6$, find the value of $\dfrac{dy}{dx}$ when $x = -1$.

(c) Find the slope and hence the equation of the tangent to the curve $y = (1-x^2)(2x-3)$ at the point $(0, -3)$.

(d) A particle moves along a straight line so that its distance s metres from a fixed point o after t seconds is given by $s = t^3 + t^2 + 2t - 5$. Find
(i) its speed after 1 second
(ii) the acceleration when $t = 2$
(iii) after how many seconds is the acceleration $8\,\text{m/sec}^2$?

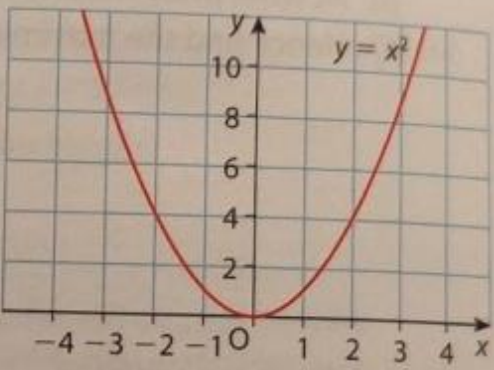
8. (a) If $y = (x^2-2)(3x-1)$, find the value of $\dfrac{dy}{dx}$ at $x = 0$.

(b) Differentiate $\dfrac{1}{x - x^2}$ with respect to x.

3. (a) Differentiate with respect to x each of these:

(i) $3x + 2$ (ii) $\frac{1}{2}x^2 - 3x - 5$ (iii) $2x^3 - x^2 - 9x$

(b) The graph of the curve $y = x^2$ is shown on the right.

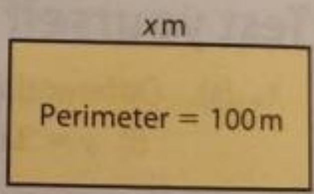


(i) Find the slope of the tangent to this curve at the point where $x = 2$.

(ii) At what point on the curve is the slope of the tangent -2?

(iii) Find the coordinates of the point on the curve at which the slope is zero.

(c) A rectangular patch of ground is to be enclosed with 100 metres of fencing wire.



(i) If the length of the patch is x metres, express the width in terms of x.

(ii) Express the area, A m^2, in terms of x.

(iii) Find the value of x for which A is a maximum.

(iv) Find this maximum area.

(10) $y = \dfrac{(x+1)^2}{x-1} \quad \begin{matrix} u \\ v \end{matrix}$

$$\frac{dy}{dx} = \frac{(x-1)2(x+1)1 - (x+1)^2 1}{(x-1)^2}$$

$$= \frac{(2x-2)(x+1) - (x^2+2x+1)}{(x-1)^2}$$

$$= \frac{2x^2 + \cancel{2x} - \cancel{2x} - 2 - x^2 - 2x - 1}{(x-1)^2}$$

$$\frac{dy}{dx} = \boxed{\frac{x^2 - 2x - 3}{(x-1)^2}} \quad \text{or} \quad \frac{(x+1)(x-3)}{(x-1)^2}$$

Remember $\dfrac{dy}{dx} = 0 \quad \therefore \quad \dfrac{(x+1)(x-3)}{(x-1)^2} = 0$

Multiple both side by $(x-1)^2$

$$\cancel{(x-1)^2}\frac{(x+1)(x-3)}{\cancel{(x-1)^2}} = 0\,(x-1)^2$$

$$(x+1)(x-3) = 0$$

$$x+1 = 0 \qquad x-3 = 0$$

$$\boxed{x = -1} \qquad \boxed{x = 3}$$

$$\frac{d^2y}{dx^2} = \frac{(x-1)^2(2x-2) - (x^2-2x-3)2(x-1)(1))}{((x-1)^2)^2}$$

Substituting $x = -1$ | Substituting $x = 3$

$$\frac{((-1)-1)^2(2(-1)-2) - ((-1)^2-2(-1)-3)2((-1)-1)(1)}{(((-1)-1)^2)^2} = \frac{-16}{16} \qquad -1 \text{ MAX}$$

$$\frac{(3-1)^2(2(3)-2) - ((3^2-2(3)-3)2((3)-1)1}{((3-1)^2)^2} = \frac{16}{16} \qquad +1 \text{ MIN}$$

To find the exact x and y coordinates find f(-1) and f(3). This will provide the y values hence you have the local maximum and local minimum points.

Maximum is (-1, 0) and minimum is (3, -8)

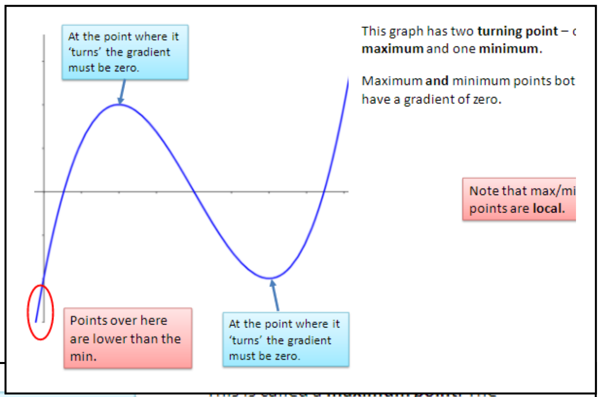
Calculus – Differentiation

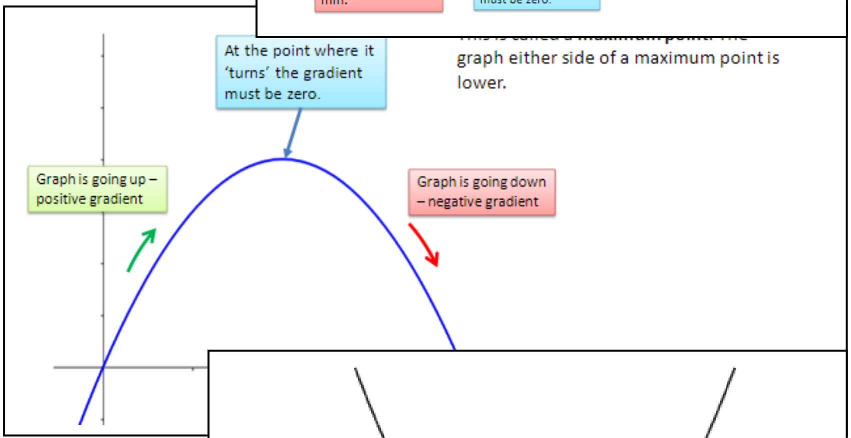
Maximum, Minimum Points of Inflection

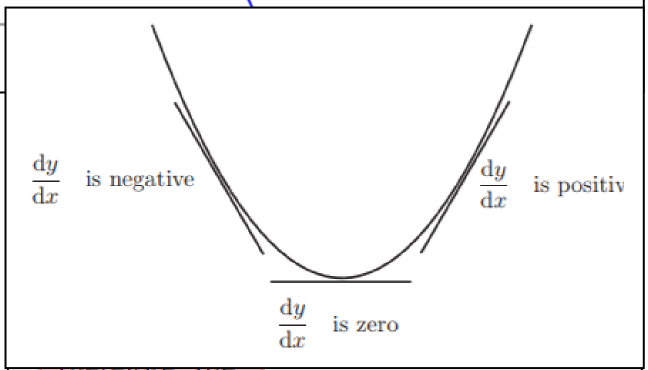
The value f '(x) is the gradient at any point but often we want to find the Turning or Stationary Point (Maximum and Minimum points) or Point of Inflection

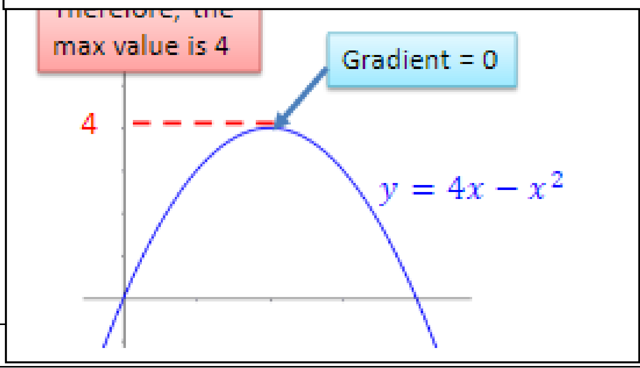
These happen where the gradient is zero, f '(x) = 0.

- f ''(x) is negative the function is maximum turning point
- f ''(x) is zero the function may be a point of inflection
- f ''(x) is positive the function is minimum turning point









Example

Find the maximum and minimum points of

- $f(x) = 2x^3 - 3x^2 - 6$

Differentiate twice as follows

- $f^I(x) = 6x^2 - 6x$
- $f^{II}(x) = \ 12x - 6$

Turning points at $f^I(x) = 0$ therefore

- $6x^2 - 6x = 0$
- $6x(x-1) = 0$
- $6x = 0 \ or \ (x-1) = 0$
- $x = 0 \ or \ x = 1$

Substituting x values into the function f(x)

- $f(0) = 2(0)^3 - 3(0)^2 - 6 \ therefore \ f(x) \ or \ y = -6.$
- $f(1) = 2(1)^3 - 3(1)^2 - 6 \ therefore \ f(x) \ or \ y = -7.$

Determining which is max/min

- $f^{II}(0) = \ 12(0) - 6 = -6$ (Maximum)
- $f^{II}(1) = \ 12(1) - 6 = \ 6$ (Minimum)

Maximum and minimum points

- Maximum turning point = $(0, -6)$
- Minimum turning point = $(1, -7)$

Exercise

1. Find the turning points of the following and determine the maximum or minimum
   - $f(x) = x^2 - 2x - 2$
   - $f(x) = x^3 - 6x^2 + 9x - 2$

2. The diagram shows a rectangular enclosure with a wall forming one side. A rope 20m long is used to form the remaining 3 sides. The width of the enclosure is x metres. Find the maximum length of x which gives the maximum area. Hence find the maximum area. You can call the length of the enclosure y.

Solution to 2 on following page(s)

More exercises *(answer on next page)*

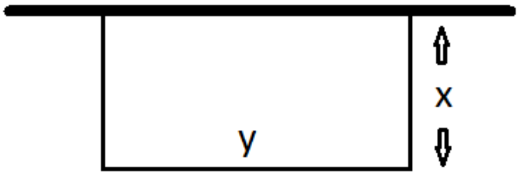
Locate the position and nature of any turning points of the following functions.

1. $y = \frac{1}{2}x^2 - 2x$, 2. $y = x^2 + 4x + 1$, 3. $y = 12x - 2x^2$, 4. $y = -3x^2 + 3x + 1$,

5. $y = x^4 + 2$, 6. $y = 7 - 2x^4$, 7. $y = 2x^3 - 9x^2 + 12x$, 8. $y = 4x^3 - 6x^2 - 72x + 1$,

9. $y = -4x^3 + 30x^2 - 48x - 1$, 10. $y = \dfrac{(x+1)^2}{x-1}$.

Solution to 2



20 = x + y + x or y =20 - 2x

Area $= xy$

Area $= x(20-2x)$

$= 20x - 2x^2$ This is the equation for graph of area against width

Maximum or minimum area occurs where $\frac{dA}{dx} = 0$

- $\frac{dA}{dx} = 20 - 4x$
- $20 - 4x = 0$
- $20 = 4x$
- $5 = x$

Maximum Area

- $\frac{d^2A}{dx^2} = -4$

Area in metre^2

$$5\big(20 - 2(5)\big) =$$
$$5(20-10) =$$
$$5(10) =$$
$$\boldsymbol{Area = 50m^2}$$

Answers to exercises on previous page

1. Minimum at $(2, -2)$,
2. Minimum at $(-2, -3)$,
3. Maximum at $(-3, -54)$,
4. Maximum at $(\frac{1}{2}, \frac{7}{4})$,
5. Minimum at $(0, 2)$,
6. Maximum at $(0, 7)$,
7. Maximum at $(1, 5)$, minimum at $(2, 4)$,
8. Maximum at $(-2, 89)$, minimum at $(3, -161)$,
9. Maximum at $(4, 31)$, minimum at $(1, -23)$,
10. Maximum at $(-1, 0)$, minimum at $(3, 8)$.

Applied Maximum and Minimum Problems

The process of finding maximum or minimum values is called **optimisation**.

We are trying to do things like maximise the profit in a company, or minimise the costs, or find the least amount of material to make a particular object.

These are very important in the world of industry.

Example 1

The daily profit, *P*, of an oil refinery is given by

$$P = 8x - 0.02x^2$$

where *x* is the number of barrels of oil refined. How many barrels will give maximum profit and what is the maximum profit?

Solution

The profit is a max (or min) if $\frac{dP}{dx} = 0$.

$$\frac{dP}{dx} = 8 - 0.04x$$

$$= 0$$

when

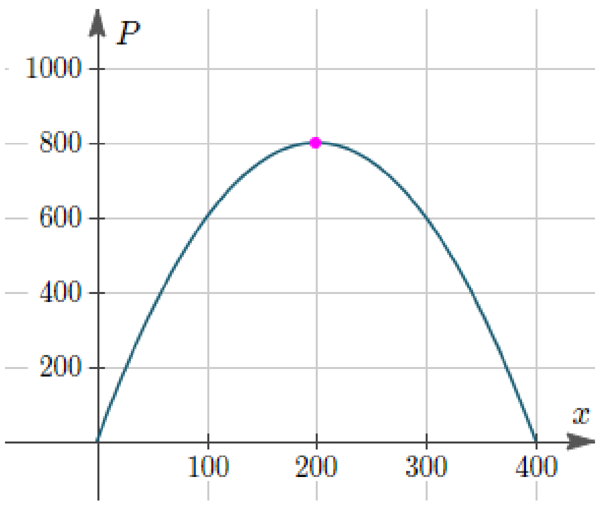
$$x = \frac{8}{0.04} = 200$$

Is it a maximum?

$\frac{d^2P}{dx^2} = -0.04 < 0$ for all x, so we have a maximum.

When $x = 200$, $P = \$800$.

So if the company refines 200 barrels per day, the maximum profit of \$800 is reached.



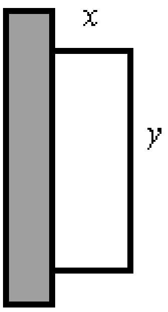
Graph of $P = 8x - 0.02x^2$.

The maximum point, $(200, 800)$ is indicated on the graph with a magenta dot.

Example 2

A rectangular storage area is to be constructed along the side of a tall building. A security fence is required along the remaining 3 sides of the area. What is the maximum area that can be enclosed with 800 m of fencing?

Solution



The area is $A = xy$

We know $2x + y = 800$ so $y = 800 - 2x$

So the area is $A = x(800 - 2x) = 800x - 2x^2$

To maximise the area, find when $\frac{dA}{dx} = 0$

$$\frac{dA}{dx} = 800 - 4x = 0$$

when

$$x = 200$$

Is it a maximum?

$$\frac{d^2A}{dx^2} = -4 < 0 \text{ for all } x$$

So it is a maximum.

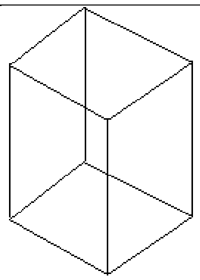
So the maximum area occurs when $x = 200$, $y = 400$ and that area is:

$A = 200 \times 400 = 80\ 000\ \mathrm{m}^2 = 8\ \mathrm{ha}$

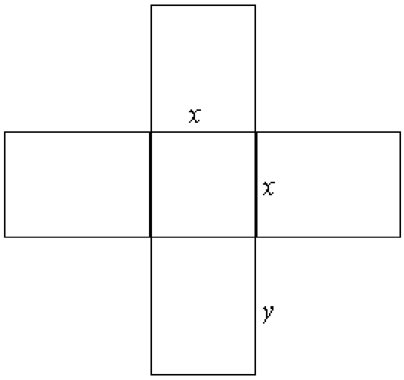
Example 3

A box with a square base has no top. If 64 cm^2 of material is used, what is the maximum possible volume for the box?

Solution



The net for this box would be:



The **volume** of the box is $V = x^2y$

We are told that the surface area of the box is $64\ \mathrm{cm}^2$. The area of the base of the box is x^2 and the area of each side is xy, so the area of the base plus the area of the 4 sides is given by:

$$x^2 + 4xy = 64\ \mathrm{cm}^2$$

Solving for y gives:

$$y = \frac{64 - x^2}{4x} = \frac{16}{x} - \frac{x}{4}$$

So the volume can be rewritten:

$$\begin{aligned} V &= x^2 y \\ &= x^2\left(\frac{16}{x} - \frac{x}{4}\right) \\ &= 16x - \frac{x^3}{4} \end{aligned}$$

Now

$$\frac{dV}{dx} = 16 - \frac{3x^2}{4}$$

and this is zero when

$$x = \pm\frac{8}{\sqrt{3}} \approx 4.62$$

(**Note:** The negative case has no practical meaning.)

Is it a maximum?

$$\frac{d^2V}{dx^2} = -\frac{3x}{2}$$

and this is negative when x is positive. So it is a MAX.

So the **dimensions** of the box are:

Base $4.62\text{ cm} \times 4.62\text{ cm}$ and sides 2.31 cm.

The **maximum possible volume** is

$V = 4.62 \times 4.62 \times 2.31 \approx 49.3\text{ cm}^3$

Check: Area of material:

$x^2 + 4xy = 21.3 + 4 \times 4.62 \times 2.31 = 64$