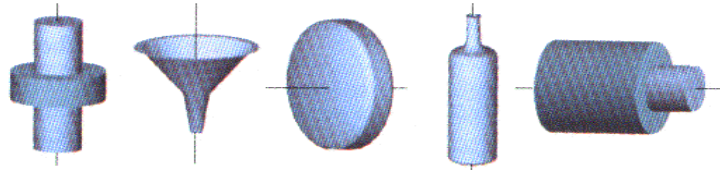# Volumes by Integration

1. **Finding volume of a solid of revolution using a disc method.**
2. **Finding volume of a solid of revolution using a washer method.**
3. **Finding volume of a solid of revolution using a shell method.**

Solids of Revolution

If a region in the plane is revolved about a given line, the resulting solid is a solid of revolution, and the line is called the axis of revolution. When calculating the volume of a solid generated by revolving a region bounded by a given function about an axis, follow the steps below:

1. Sketch the area and determine the axis of revolution, (this determines the variable of integration)
2. Sketch the cross-section, (disk, shell, washer) and determine the appropriate formula.
3. Determine the boundaries of the solid,
4. Set up the definite integral, and integrate.

1. Finding volume of a solid of revolution using a disc method.

The simplest solid of revolution is a right circular cylinder which is formed by revolving a rectangle about an axis adjacent to one side of the rectangle, (the disc).

To see how to calculate the volume of a general solid of revolution with a disc cross-section, using integration techniques, consider the following solid of revolution formed by revolving the plane region bounded by *f(x)*, y-axis and the vertical line x=2 about the x-axis. (see Figure1 to 4 below):

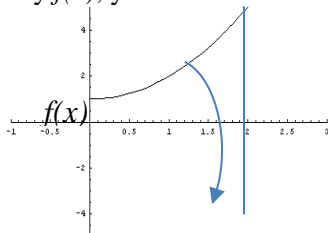Figure 1.   The area under *f(x),* bounded by *f(x)*, x-axis, y-axis and the vertical line x=2 rotated about x-axis
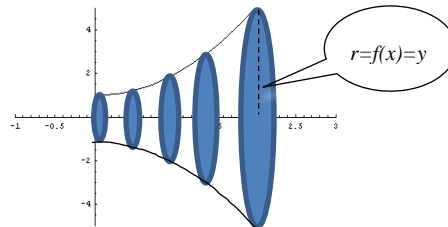
Figure 2. Basic sketch of the solid of revolution with few typical discs indicated.
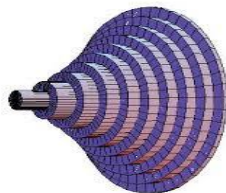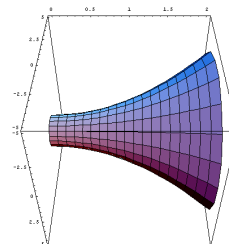
Figure 3. Family of discs

Figure 4. The 3-D model of the solid of revolution.

FORMULAS: $V = \int A\,dx$, or respectively $\int A\,dy$ where A stands for the area of the typical disc.

Another words: $A = \pi r^2$ and $r=f(x)$ or $r=f(y)$ depending on the axis of revolution.

1. The volume of the solid generated by a region under $f(x)$ bounded by the x-axis and vertical lines $x=a$ and $x=b$, which is revolved **about the x-axis** is

$$V = \pi \int_a^b y^2\,dx = \pi \int_a^b \left[f(x)\right]^2 dx \qquad \text{(disc with respect to x and } r=y=f(x))$$
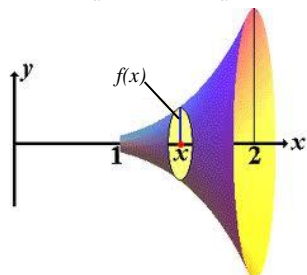
2. The volume of the solid generated by a region under $f(y)$ (to the left of $f(y)$ bounded by the y-axis, and horizontal lines $y=c$ and $y=d$ which is revolved **about the y-axis**.

$$V = \pi \int_c^d x^2\,dy = \pi \int_c^d \left[f(y)\right]^2 dy \qquad \text{(disc with respect to y and } r=x=f(y))$$
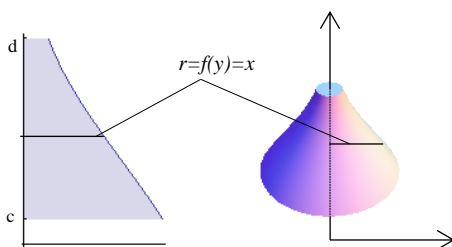
Ex. 1. (Source: **Paul Dawkins**) http://tutorial.math.lamar.edu/Classes/CalcI/VolumeWithRings.aspx

Determine the volume of the solid generated by rotating the region bounded by $f(x) = x^2 - 4x + 5$, $x = 1$, $x = 4$ and the x-axis about the x-axis.

Solution:
Step 1 is to sketch the bounding region and the solid obtained by rotating the region about the x-axis. Here are both of these sketches.
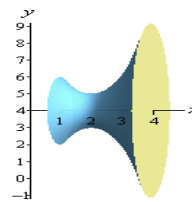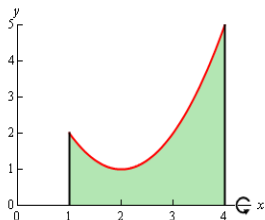
Step 2: To get a cross section we cut the solid at any x, since the x-axis it the axis of rotation.

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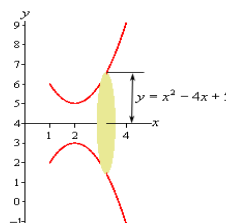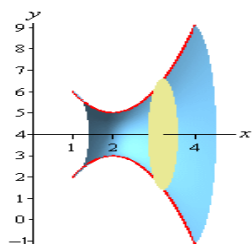
In this case the radius is simply the distance from the *x*-axis to the curve and this is nothing more than the function value at that particular *x* as shown above. The cross-sectional area is the $A(x) = \pi r^2 = \pi[f(x)]^2$ which in this case is equal to $A(x) = \pi(x^2 - 4x + 5)^2 = \pi(x^4 - 8x^3 + 26x^2 - 40x + 25)$

Step3. Determine the boundaries which will represent the limits of integration. Working from left to right the first cross section will occur at $x = 1$, and the last cross section will occur at $x = 4$. These are the limits of integration.

Step 4. Integrate to find the volume:

$$V = \int_a^b A(x)dx = \pi\int_a^b f(x)dx = \pi\int_1^4 (x^2 - 4x + 5)^2 dx = \pi\int_1^4 (x^4 - 8x^3 + 26x^2 - 40x + 25)dx =$$

$$\pi\left(\frac{1}{5}x^5 - 2x^4 + \frac{26}{3}x^3 - 20x^2 + 25x\right)\Big|_1^4 = \frac{78\pi}{5}$$

The volume of the solid generated by rotating the region bounded by $f(x) = x^2 - 4x + 5$, $x = 1$, $x = 4$ and the *x*-axis about the *x*-axis is $\frac{78\pi}{5}$ units cubed.

2. Finding volume of a solid of revolution using a washer method.

This is an extension of the disc method. The procedure is essentially the same, but now we are dealing with a hollowed object and two functions instead of one, so we have to take the difference of these functions into the account.

The general formula in this case would be:
$$A = \pi(R^2 - r^2)$$ where R is an outer radius and r is the inner radius.

FORMULAS: $V = \int A(x)dx$, or respectively $\int A(y)dy$

1. The volume of the solid generated by a region between *f(x)* and *g(x)* bounded by the vertical lines x=a and x=b, which is revolved about the x-axis is

$$V = \pi\int_a^b |(f(x))^2 - (g(x))^2| dx \qquad \text{(washer with respect to x)}$$

2. The volume of the solid generated by a region between *f(y)* and *g(y)* bounded by the horizontal lines y=c and y=d which is revolved about the y-axis.

$$V = \pi\int_c^d |(f(y))^2 - (g(y))^2| dy \qquad \text{(washer with respect to y)}$$

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Example 2 (Source: **Paul Dawkins**) http://tutorial.math.lamar.edu/Classes/CalcI/VolumeWithRings.aspx

Determine the volume of the solid generated by rotating the region bounded by $y = \sqrt[3]{x}$, and $y = \dfrac{x}{4}$ that lies in the first quadrant about the *y*-axis.

Solution
Step 1: Graph the bounding region and a graph of the object. The cross section is cut **perpendicular** to the axis of rotation and it is a horizontal washer. The inner and outer radii of the washer are x values, so we will need to rewrite our functions into the form $x = f(y)$.
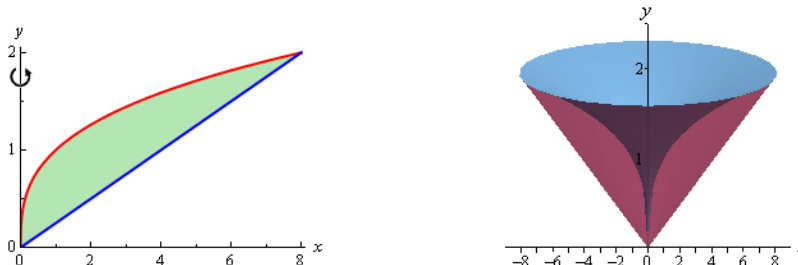
Here are the functions written in the correct form for this example.

$$y = \sqrt[3]{x} \Rightarrow x = y^3 \quad \text{and} \quad y = \frac{x}{4} \Rightarrow x = 4y$$

Step 2. Graph couple of sketches of the boundaries of the walls of this object as well as a typical washer. The sketch on the left includes the back portion of the object to give a little context to the figure on the right.
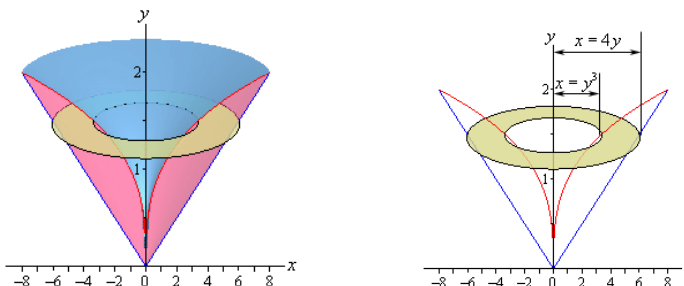
The cross-sectional area is then, $A(y) = \pi\left((4y)^2 - (y^3)^2\right) = \pi\left(16y^2 - y^6\right)$

Step 3. Working from the bottom of the solid to the top we can see that the first cross-section will occur at *y=0* and the last cross-section will occur at *y=2*. These will be the limits of integration.

Step 4. The volume is then, $V = \displaystyle\int_c^d A(y)\,dy = \pi\int_0^2 \left(16y^2 - y^6\right)dy = \pi\left(\dfrac{16}{3}y^3 - \dfrac{1}{7}y^7\right)\Bigg|_0^2 = \dfrac{512\pi}{21}$

We can obtain the solids by rotating the given regions about any line; x- and y-axes are just the simpler cases. The next example the solids of revolution can be obtained by rotating about a given horizontal line.

*Example 3* (Source: **Paul Dawkins**) http://tutorial.math.lamar.edu/Classes/CalcI/VolumeWithRings.aspx

Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$ and $y = x$ about the line $y = 4$.

Solution
Step1. Graph the bounding region and the solid. Also, since we are rotating about a horizontal axis, we know that the cross-sectional area will be a function of *x*.

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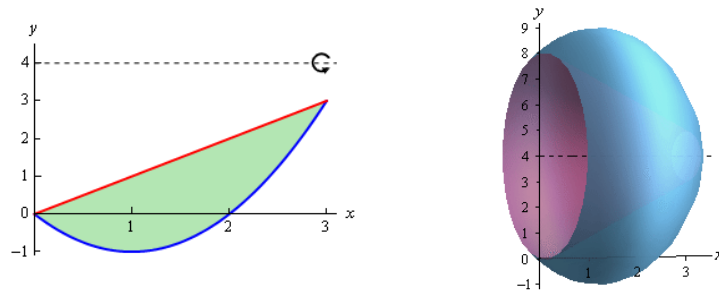
Here are a couple of sketches of the boundaries of the walls of this object as well as a typical washer.
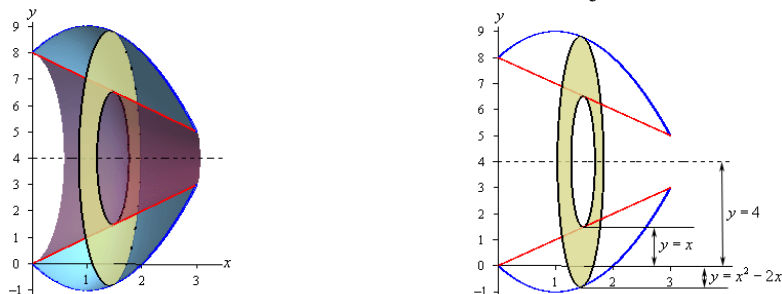
Step 2. Now, we're going to have to be careful here in determining the inner and outer radii as they aren't going to be quite as simple they were in the previous two examples.

Let's start with the inner radius as this one is a little clearer. First, the inner radius is NOT x. The distance from the x-axis to the inner edge of the washer is x, but we want the radius and that is the distance from the axis of rotation to the inner edge of the washer. So, we know that the distance from the axis of rotation to the x-axis is 4 and the distance from the x-axis to the inner washer is x. The inner radius must then be the difference between these two. Or: **inner radius** $= 4 - x$

The outer radius works the same way. The **outer radius** $= 4 - \left(x^2 - 2x\right) = -x^2 + 2x + 4$

Note that given the location of the typical washer in the sketch above the formula for the outer radius may not look quite right but it is in fact correct. As sketched, the outer edge of the washer is below the x-axis and at this point the value of the function will be negative and so when we do the subtraction in the formula for the outer radius we'll actually be subtracting off a negative number which has the net effect of adding this distance onto 4 and that gives the correct outer radius.

The cross-sectional area for this case is:
$$A(x) = \pi\left(\left(-x^2 + 2x + 4\right)^2 - \left(4 - x\right)^2\right) = \pi\left(x^4 - 4x^3 - 5x^2 + 24x\right)$$

Step 3. The lower boundary will be at $x = 0$ and the upper boundary will occur at $x = 3$ and so these are our limits of integration.

Step 4. The volume is then
$$V = \int_a^b A(x)\,dx = \pi\int_0^3 \left(x^4 - 4x^3 - 5x^2 + 24x\right)dx = \pi\left(\frac{1}{5}x^5 - x^4 - \frac{5}{3}x^3 + 12x^2\right)\bigg|_0^3 = \frac{153}{5}\pi$$

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Similar procedure applies when the region is rotated about a vertical line $x = a$. In this case, similarly to example 2, the working variable will be y (integral will be set up with respect to y, and the radii need to be adjusted by taking the shift $x = a$ into account).

## 3. Finding volume of a solid of revolution using a shell method.

The shell method is a method of calculating the volume of a solid of revolution when integrating along an axis parallel to the axis of revolution. It is less intuitive than disk integration, but it usually produces simpler integrals. It makes use of the so-called "representative cylinder" when the part of the graph of a function is rotated around an axis, and is modeled by an infinite number of hollow pipes, all infinitely thin. The idea is that a "representative rectangle" can be rotated about the axis of revolution, thus generating a hollow cylinder – a shell. Volume of the solid is then calculated by integrating the lateral surface areas of the "family" of shells
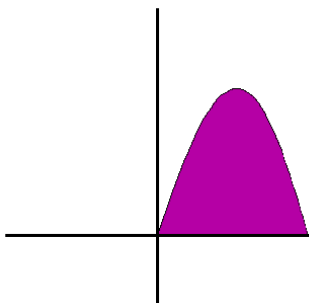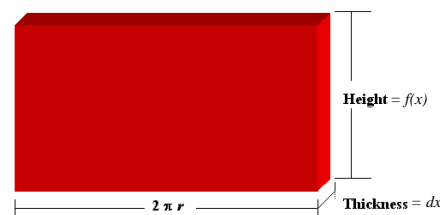

The region to be revolved around y axis
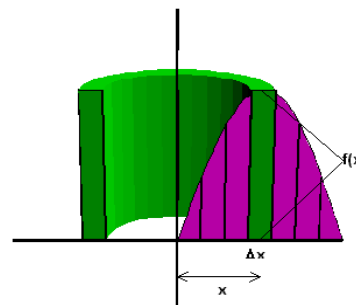
Region with a representative rectangle rotated half way
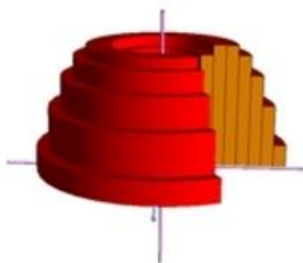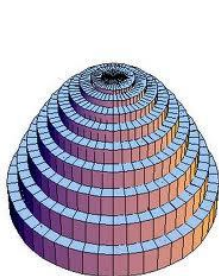
Family of shells

Dimensions of a typical shell

FORMULAS: V= $\int length \cdot height \cdot thickness$, or respectively $\int 2\pi \cdot radius \cdot height \cdot thickness$

1. The volume of the solid generated by a region bounded by the vertical lines $x=a$ and $x=b$, which is revolved about the y-axis (vertical axis of revolution so the radius is equal to x, (in graphs above) is $2\pi \int_{a}^{b} xf(x)dx$, and radius $r=x$

2. The volume of the solid generated by a region bounded by the y-axis, and horizontal lines $y=c$ and $y=d$ which is revolved about the x-axis (horizontal axis of revolution) is $2\pi \int_{c}^{d} yf(y)dy$

    and radius $r=y$

R·I·T

Example 4. (Source : **Rafael Espericueta**)
http://www2.bc.cc.ca.us/resperic/Math6A/Lectures/ch6/2/shell.htm

Find the volume of the solid of revolution formed by rotating the finite region bounded by the graphs of $y = \sqrt{x-1}$ and $y = (x-1)^2$ about the *y*-axis

Once you visualize the shells, the volume integral is almost in your grasp!
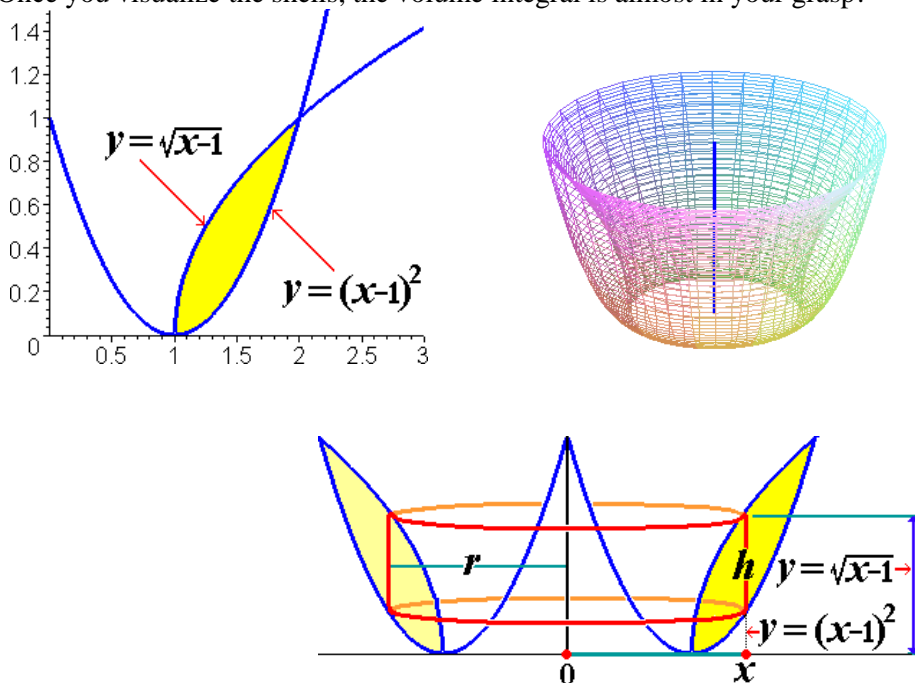
This time, the height of the cylindrical shell is the difference of the two functions, so the volume of an individual shell will be given by $V = 2\pi r h dx = 2\pi x\left(\sqrt{x-1} - (x-1)^2\right)dx$

Integrating this over the interval *x=1* to *x=2*, we obtain the volume of the solid of revolution:

$$V = 2\pi \int_1^2 x\left(\sqrt{x-1} - (x-1)^2\right)dx = 2\pi \int_1^2 \left(x\sqrt{x-1} - x(x-1)^2\right)dx$$

Use u-substitution to integrate the given function. Set *u=x-1* so *du=dx* and *x=u+1*. Don't forget to adjust the boundaries accordingly, so the integral becomes:

$$V = 2\pi \int_0^1 \left((u+1)\sqrt{u} - (u+1)u^2\right)du = 2\pi \int_0^1 \left(u\sqrt{u} + \sqrt{u} - u^3 - u^2\right)du = 2\pi\left(\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} - \frac{1}{4}u^4 - \frac{1}{3}u^3\right)\Bigg|_0^1 = \frac{29}{30}\pi$$

You try it: Find the volume of a solid generated by rotating:

1.   First quadrant region bounded by $y = (x-1)(x-3)^2$, $y = 0$ rotated about the y-axis
2.   Region bounded by $y = \sqrt{x}$, $y = 4$, $x = 0$ rotated about the y-axis
3.   Region bounded by $y = x^2 + 2$, $y = x + 4$ rotated about the x-axis

R·I·T

Solutions:

1. $\dfrac{24}{5}\pi$ (shell)

2. $\dfrac{1024}{5}\pi$ (disk)

3. $\dfrac{162}{5}\pi$ (washer)

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