10

The Quadratic Function

TERMINOLOGY

Axis of symmetry: A line about which two parts of a graph are symmetrical. One half of the graph is a reflection of the other

Coefficient: A constant multiplied by a pronumeral in an algebraic term e.g. in ax^3 the a is the coefficient

Discriminant: Part of the quadratic formula, the algebraic expression $b^2 - 4ac$ is called the discriminant as its value determines the number and nature of the roots of a quadratic equation

Equations reducible to quadratics: Equations that can be reduced to the form: $ax^2 + bx + c = 0$

Indefinite: A quadratic function where $f(x)$ can be both positive and negative for varying values of x

Maximum value: The maximum or greatest y-value of a graph for a given domain

Minimum value: The minimum or smallest y-value of a graph for a given domain

Negative definite: A quadratic function where $f(x)$ is always negative for all values of x

Positive definite: A quadratic function where $f(x)$ is always positive for all values of x

Root of an equation: The solution of an equation

INTRODUCTION

THE SOLUTION OF QUADRATIC equations is important in many fields, such as engineering, architecture and astronomy. In this chapter you will study quadratic equations in detail, and look at the relationship between quadratic equations and the graphs of quadratic functions (the parabola). You will study the **axis of symmetry** and **maximum and minimum values** of the quadratic function. You will also look at the quadratic formula in detail, and at the relationships between the **roots** (solutions) of quadratic equations, the formula and the quadratic function.

DID YOU KNOW?

Thousands of clay tablets from ancient Babylonia have been discovered by archaeologists. These tablets are from as far back as 2000 BC. They show that the Babylonians had mastered many mathematical skills. Geometry, including Pythagoras' theorem, was well developed, and geometric problems were often worked out by using algebra.

Quadratic equations were used in solving geometry problems. The word 'quadratic' comes from the Latin '*quadratum*', meaning 'four-sided figure'. Completing the square and the quadratic formula were both used to solve quadratic equations.

The Babylonians also had some interesting approximations for square roots. For example, $\sqrt{2} = \frac{17}{12}$. An approximation for $\sqrt{2}$ that is very accurate was found on a tablet dating back to 1600 BC:

$$\sqrt{2} = 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} = 1.414213$$

Graph of a Quadratic Function

Axis of symmetry

EXAMPLE

(a) Sketch the parabola $y = x^2 - 4x$ on the number plane.
(b) Find the equation of the axis of symmetry of the parabola.
(c) Find the minimum value of the parabola.

Solution

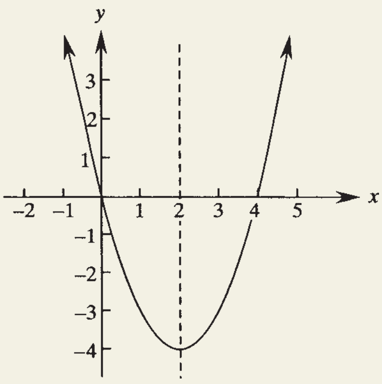
(a) For the y-intercept, $x = 0$

i.e. $y = 0^2 - 4(0)$
$= 0$

For the x-intercept, $y = 0$

i.e. $0 = x^2 - 4x$
$= x(x - 4)$

$\therefore\ x = 0$ or $x - 4 = 0$
$x = 4$



The axis of symmetry lies halfway between $x = 0$ and $x = 4$.

CONTINUED

(b) The axis of symmetry has equation $x = 2$.
(c) Since the parabola is symmetrical about the line $x = 2$, the minimum value is on this line.

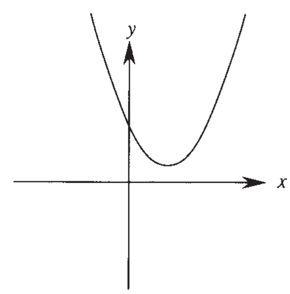
Substitute $x = 2$ into the equation of the parabola

i.e. $y = 2^2 - 4(2)$
$= -4$

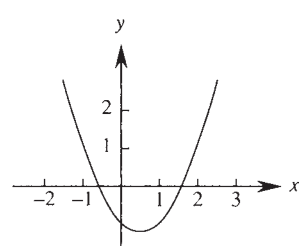
So the minimum value is -4.

Class Investigation

1. How would you find the axis of symmetry for a graph with no x-intercepts?



2. How would you find the axis of symmetry of a graph where the x-intercepts are irrational numbers?



The axis of symmetry of the quadratic function $y = ax^2 + bx + c$ has the equation

$$x = -\frac{b}{2a}$$

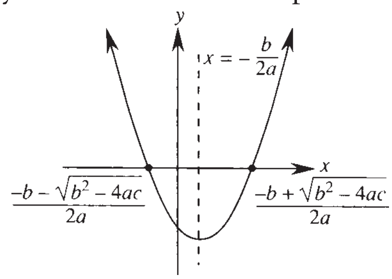
Proof

The axis of symmetry lies midway between the x-intercepts.

For the x-intercepts, $y = 0$

i.e. $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



The x-coordinate of the axis of symmetry is the average of the x-intercepts.

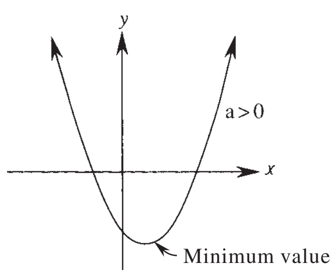
$$\text{i.e.} \quad x = \frac{\frac{-b - \sqrt{b^2 - 4ac}}{2a} + \frac{-b + \sqrt{b^2 - 4ac}}{2a}}{2}$$

$$= \frac{\frac{-2b}{2a}}{2}$$

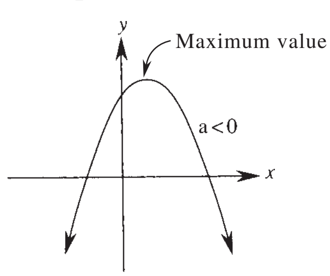
$$= \frac{-2b}{4a}$$

$$= -\frac{b}{2a}$$

The parabola has a minimum value if $a > 0$.
The shape of the parabola is concave upwards.



The parabola has a maximum value if $a < 0$.
The shape of the parabola is concave downwards.



The minimum or maximum value is $f\left(-\frac{b}{2a}\right)$

EXAMPLES

1. Find the equation of the axis of symmetry and the minimum value of the quadratic function $y = x^2 - 5x + 1$.

Solution

The equation of the axis of symmetry is given by

$$x = -\frac{b}{2a}$$

i.e.
$$x = -\frac{(-5)}{2(1)} = \frac{5}{2}$$

$\therefore$ Equation is $x = 2\frac{1}{2}$

Minimum value:
$$y = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) + 1 = \frac{25}{4} - \frac{25}{2} + 1 = -5\frac{1}{4}$$

$a > 0$ gives a minimum value.

So minimum value is $-5\frac{1}{4}$.

---

2. Find the equation of the axis of symmetry and the maximum value of the quadratic function $y = -3x^2 + x - 5$.

Solution

The equation of the axis of symmetry is given by

$$x = -\frac{b}{2a}$$

i.e.
$$x = -\frac{1}{2(-3)} = \frac{1}{6}$$

$\therefore$ Equation is $x = \frac{1}{6}$

Maximum value:
$$y = -3\left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right) - 5 = -\frac{1}{12} + \frac{1}{6} - 5 = -4\frac{11}{12}$$

$a < 0$ gives a maximum value.

So maximum value is $-4\frac{11}{12}$.

Class Investigation

Examine the graph of $y = -3x^2 + x - 5$ from the above example. Are there any solutions for the quadratic equation $-3x^2 + x - 5 = 0$?

The minimum or maximum point of the parabola is called the vertex.

EXAMPLE

(a) Find the equation of the axis of symmetry and the coordinates of the vertex of the parabola $y = 2x^2 - 12x + 7$.
(b) Find the y-intercept and sketch the graph.

Solution

(a) Axis of symmetry:

$$x = -\frac{b}{2a}$$
$$= -\frac{-12}{2 \times 2}$$
$$= 3$$

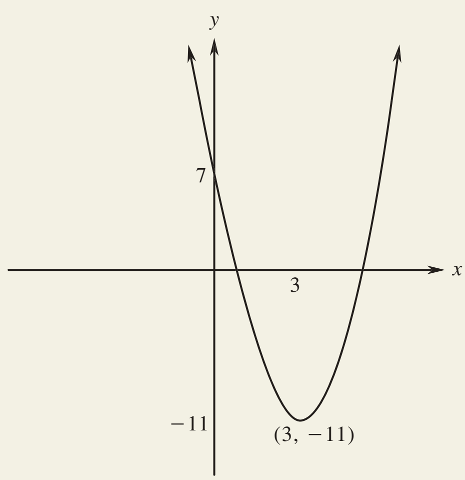
When $x = 3$

$$y = 2(3)^2 - 12(3) + 7$$
$$= -11$$

So the vertex is $(3, -11)$.

(b) For y-intercept, $x = 0$

$$y = 2(0)^2 - 12(0) + 7$$
$$= 7$$



The vertex is the minimum point of the parabola since $a > 0$.

10.1 Exercises

1. By finding the intercepts on the axes, sketch the parabola $y = x^2 + 2x$. Find the equation of its axis of symmetry, and the minimum value.

2. Find the equation of the axis of symmetry and the minimum value of the parabola $y = 2x^2 + 6x - 3$.

3. Find the equation of the axis of symmetry and the minimum value of the parabola $y = x^2 + 3x + 2$.

4. Find the equation of the axis of symmetry and the minimum value of the parabola $y = x^2 - 4$.

5. Find the equation of the axis of symmetry and the minimum point of the parabola $y = 4x^2 - 3x + 1$.

6. Find the equation of the axis of symmetry and the maximum value of the parabola $y = -x^2 + 2x - 7$.

7. Find the equation of the axis of symmetry and the maximum point of the parabola $y = -2x^2 - 4x + 5$.

8. Find the minimum value of $y = x^2 + 4x + 3$. How many solutions does the equation $x^2 + 4x + 3 = 0$ have?

9. Find the minimum value of $y = x^2 + x + 4$. How many solutions does the equation $x^2 + x + 4 = 0$ have?

10. Find the minimum value of $y = x^2 + 4x + 4$. How many solutions does the equation $x^2 + 4x + 4 = 0$ have?

11. Find the equation of the axis of symmetry and the coordinates of the vertex for each parabola.
    (a) $y = x^2 + 6x - 3$
    (b) $y = -x^2 - 8x + 1$
    (c) $y = -2x^2 + 5x$
    (d) $y = 4x^2 + 10x - 7$
    (e) $y = 3x^2 + 18x + 4$

12. Find
    (i) the equation of the axis of symmetry
    (ii) the minimum or maximum value and
    (iii) the vertex of the parabola.
    (a) $y = x^2 + 2x - 2$
    (b) $y = -2x^2 + 4x - 1$

13. Find the maximum or minimum point for each function.
    (a) $y = x^2 + 2x + 1$
    (b) $y = x^2 - 8x - 7$
    (c) $f(x) = x^2 + 4x - 3$
    (d) $y = x^2 - 2x$
    (e) $f(x) = x^2 - 4x - 7$
    (f) $f(x) = 2x^2 + x - 3$
    (g) $y = -x^2 - 2x + 5$
    (h) $y = -2x^2 + 8x + 3$
    (i) $f(x) = -3x^2 + 3x + 7$
    (j) $f(x) = -x^2 + 2x - 4$

14. For each quadratic function
    (i) find any x-intercepts using the quadratic formula.
    (ii) state whether the function has a maximum or minimum value and find this value.
    (iii) sketch the function on a number plane.
    (a) $f(x) = x^2 + 4x + 4$
    (b) $f(x) = x^2 - 2x - 3$
    (c) $y = x^2 - 6x + 1$
    (d) $f(x) = x^2 + 2x$
    (e) $y = 2x^2 - 18$
    (f) $y = 3x^2 + x - 2$
    (g) $f(x) = -x^2 - 2x + 6$
    (h) $f(x) = -x^2 - x + 3$
    (i) $y = -x^2 - 3x + 2$
    (j) $y = -2x^2 + 4x + 5$

15. (a) Find the minimum value of the parabola $y = x^2 - 2x + 5$.
    (b) How many solutions does the quadratic equation $x^2 - 2x + 5 = 0$ have?
    (c) Sketch the parabola.

16. (a) How many x-intercepts has the quadratic function $f(x) = x^2 - 3x + 9$?
    (b) Find the minimum point of the function.
    (c) Sketch the function.

17. (a) Find the maximum value of the quadratic function $f(x) = -2x^2 + x - 4$.
    (b) How many solutions has the quadratic equation $-2x^2 + x - 4 = 0$?
    (c) Sketch the graph of the quadratic function.

18. (a) Sketch the parabola $y = x^2 - 5x + 6$.
(b) From the graph, find values of x for which $x^2 - 5x + 6 > 0$.
(c) Find the domain over which $x^2 - 5x + 6 \leq 0$.

19. Sketch $y = 3x^2 - 2x + 4$ and hence show that $3x^2 - 2x + 4 > 0$ for all x.

20. By sketching $f(x) = x^2 + x + 2$, show that $x^2 + x + 2 > 0$ for all x.

21. Show by a sketch that $-x^2 + 2x - 7 < 0$ for all x.

22. Sketch $y = -5x^2 + 4x - 1$ and show that $-5x^2 + 4x - 1 < 0$ for all x.

Investigation

Could you tell without sketching the function $y = x^2 - x + 5$ if $x^2 - x + 5 > 0$ for all x? How could you do this?

How could you know that $-x^2 + 2x - 7 < 0$ for all x without sketching the graph of $f(x) = -x^2 + 2x - 7$?

You will look at this later on in the chapter.

Quadratic Inequalities

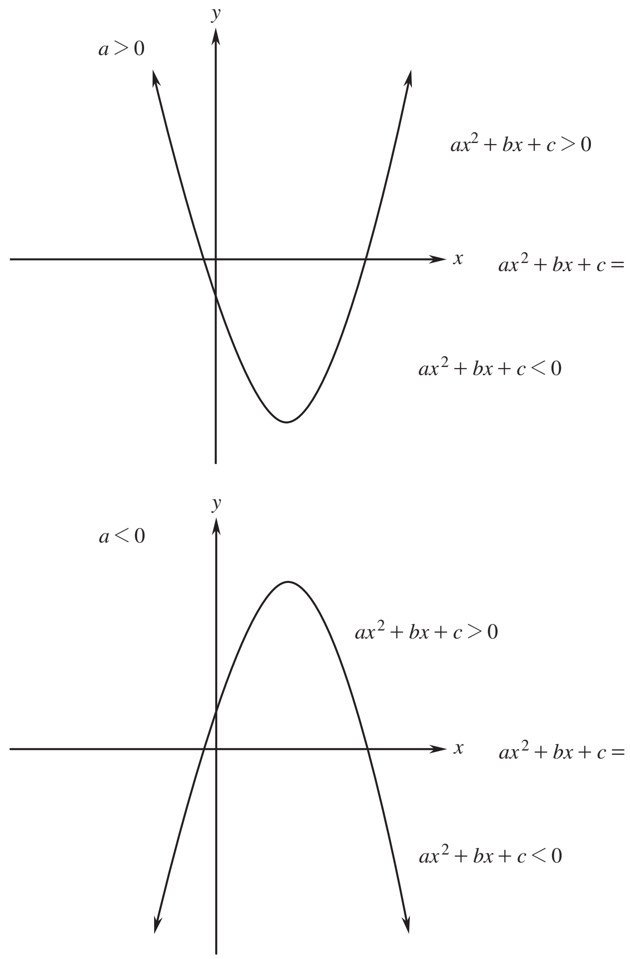
You looked at solving quadratic inequations in Chapter 3 using the number line. You can also solve them using the graph of a parabola.

For any curve on a number plane

$y = 0$ is on the x-axis	(all values of y are zero on the x-axis)
$y > 0$ is above the x-axis	(all positive values of y lie above the x-axis)
$y < 0$ is below the x-axis	(all negative values of y lie below the x-axis)

Substituting $ax^2 + bx + c$ for y in the general parabola $y = ax^2 + bx + c$ gives the following results:

For the parabola $y = ax^2 + bx + c$
$ax^2 + bx + c = 0$ on the x-axis
$ax^2 + bx + c > 0$ above the x-axis
$ax^2 + bx + c < 0$ below the x-axis



EXAMPLES

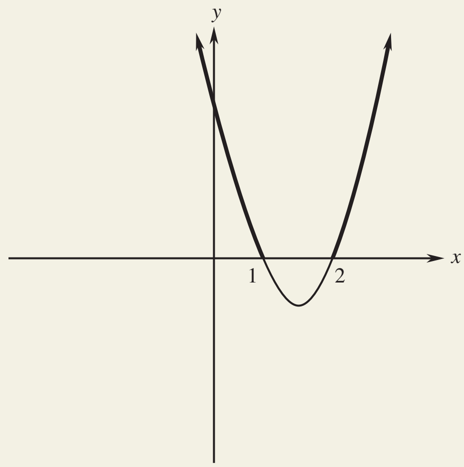
1. Solve $x^2 - 3x + 2 \geq 0$.

Solution

First sketch $y = x^2 - 3x + 2$ showing x-intercepts ($a > 0$ so it is concave upwards).

For x-intercepts, $y = 0$

$$0 = x^2 - 3x + 2$$
$$= (x - 2)(x - 1)$$
$$x - 2 = 0, \quad x - 1 = 0$$
$$x = 2, \quad x = 1$$



$y \geq 0$ on and above the x-axis
So $x^2 - 3x + 2 \geq 0$ on and above the x-axis.
$\therefore\ x \leq 1, x \geq 2$

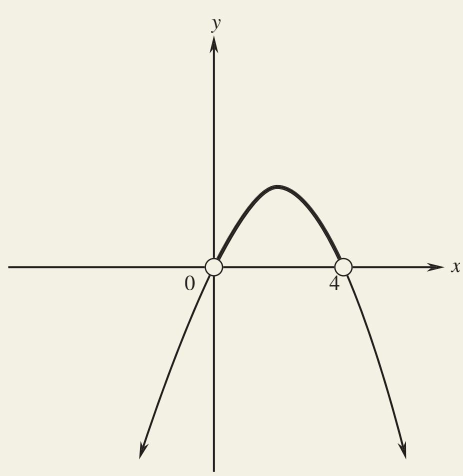
---

2. Solve $4x - x^2 > 0$.

Solution

First sketch $y = 4x - x^2$ showing x-intercepts ($a < 0$ so it is concave downwards).
For x-intercepts, $y = 0$

$$0 = 4x - x^2$$
$$= x(4 - x)$$
$$x = 0, \quad 4 - x = 0$$
$$x = 0, \quad 4 = x$$



$y > 0$ above the x-axis
So $4x - x^2 > 0$ above the x-axis.
$\therefore\ 0 < x < 4$.

CONTINUED

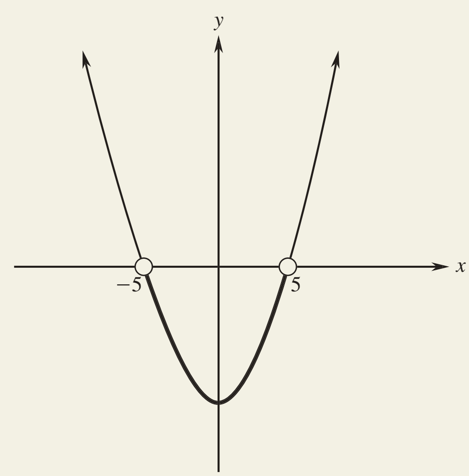
3. Solve $x^2 - 25 < 0$.

Solution

First sketch $y = x^2 - 25$ showing x-intercepts ($a > 0$ so it is concave upwards).

For x-intercepts, $y = 0$

$$0 = x^2 - 25$$
$$= (x + 5)(x - 5)$$
$$x + 5 = 0, \quad x - 5 = 0$$
$$x = -5, \quad x = 5$$



$y < 0$ below the x-axis

So $x^2 - 25 < 0$ below the x-axis.

$\therefore \ -5 < x < 5$

Further inequations

EXTENSION

You learned how to solve inequations involving the pronumeral in the denominator by using the number line in Chapter 3. Here we use quadratic inequalities to solve them.

EXAMPLES

1. Solve $\frac{1}{x+1} \ge 2$.

Solution

$x \ne -1$

We don't know whether $x + 1$ is positive or negative, but $(x + 1)^2$ is always positive. We can multiply both sides of the inequation by $(x + 1)^2$ without changing the inequality sign.

$$\frac{1}{x+1} \ge 2$$

$$\frac{1}{x+1} \times (x+1)^2 \ge 2 \times (x+1)^2$$

$$x + 1 \ge 2(x+1)^2$$

$$0 \ge 2(x+1)^2 - (x+1)$$

$$\ge (x+1)[2(x+1) - 1]$$

$$\ge (x+1)(2x + 2 - 1)$$

$$\ge (x+1)(2x+1)$$

Factorise by taking out $x + 1$ as a common factor.

We solve this by sketching the parabola $y = (x + 1)(2x + 1)$.

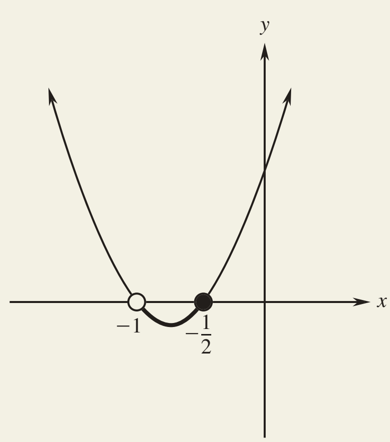
For x-intercepts: $y = 0$

$0 = (x + 1)(2x + 1)$

$x + 1 = 0, \quad 2x + 1 = 0$

$x = -1, \quad 2x = -1$

$x = -\frac{1}{2}$



$0 \ge (x + 1)(2x + 1)$ on and below the x-axis. However, $x \ne -1$

The solution is $-1 < x \le -\frac{1}{2}$.

---

2. Solve $\frac{4x}{x-2} < 5$.

Solution

$x \ne 2$

We multiply both sides of the inequation by $(x - 2)^2$.

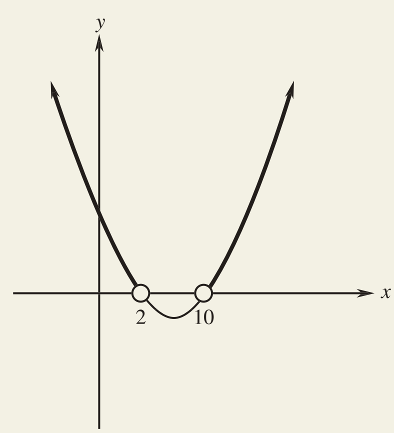
CONTINUED

$$\frac{4x}{x-2} < 5$$
$$\frac{4x}{x-2} \times (x-2)^2 < 5 \times (x-2)^2$$
$$4x(x-2) < 5(x-2)^2$$
$$0 < 5(x-2)^2 - 4x(x-2)$$
$$< (x-2)[5(x-2) - 4x]$$
$$< (x-2)(5x - 10 - 4x)$$
$$< (x-2)(x-10)$$

Factorise by taking out $x - 2$ as a common factor.

We solve this by sketching the parabola $y = (x-2)(x-10)$.
For x-intercepts: $y = 0$

$$0 = (x-2)(x-10)$$
$$x - 2 = 0, \quad x - 10 = 0$$
$$x = 2, \quad x = 10$$



$0 < (x-2)(x-10)$ above the x-axis.
The solution is $x < 2, x > 10$.

10.2 Exercises

Solve

1. $x^2 - 9 > 0$
2. $n^2 + n \le 0$
3. $a^2 - 2a \ge 0$
4. $4 - x^2 < 0$
5. $y^2 - 6y \le 0$
6. $2t - t^2 > 0$
7. $x^2 + 2x - 8 > 0$
8. $p^2 + 4p + 3 \ge 0$
9. $m^2 - 6m + 8 > 0$
10. $6 - x - x^2 \le 0$

11. $2h^2 - 7h + 6 < 0$

12. $x^2 - x - 20 \leq 0$

13. $35 + 9k - 2k^2 \geq 0$

14. $q^2 - 9q + 18 > 0$

15. $(x + 2)^2 \geq 0$

16. $12 - n - n^2 \leq 0$

17. $x^2 - 2x < 15$

18. $-t^2 \geq 4t - 12$

19. $3y^2 > 14y + 5$

20. $(x - 3)(x + 1) \geq 5$

21. $\frac{1}{x} < -2$

22. $\frac{1}{x} > 3$

23. $\frac{1}{x} \geq 1$

24. $-\frac{1}{x} \geq 2$

25. $\frac{1}{x - 1} > 3$

26. $\frac{1}{x + 2} \leq 1$

27. $\frac{2}{x - 2} \geq 5$

28. $\frac{3}{x + 3} > -1$

29. $\frac{-1}{x - 1} \leq 3$

30. $\frac{x}{x + 2} \geq 4$

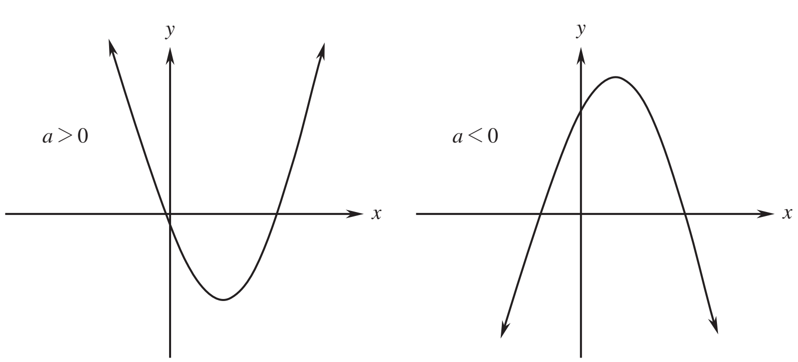
Solve the inequations in Chapter 3 using these methods for extra practice.

The Discriminant

The values of x that satisfy a quadratic equation are called the **roots** of the equation.

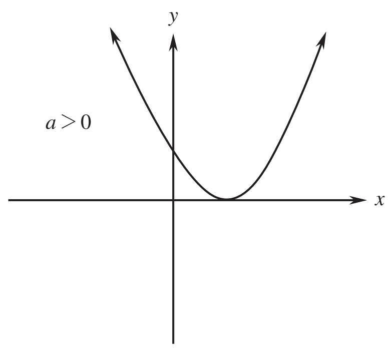
> The roots of $ax^2 + bx + c = 0$ are the x-intercepts of the graph $y = ax^2 + bx + c$

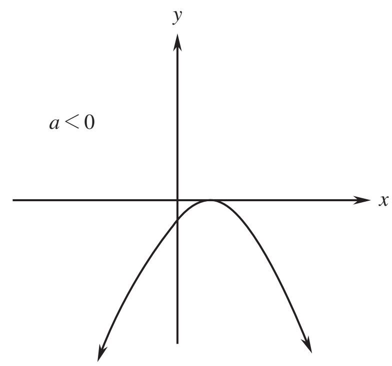
1. If $y = ax^2 + bx + c$ has 2 x-intercepts, then the quadratic equation $ax^2 + bx + c = 0$ has 2 real roots.



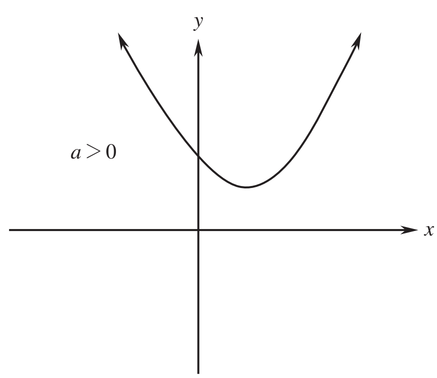
Since the graph can be both positive and negative, it is called an **indefinite** function.

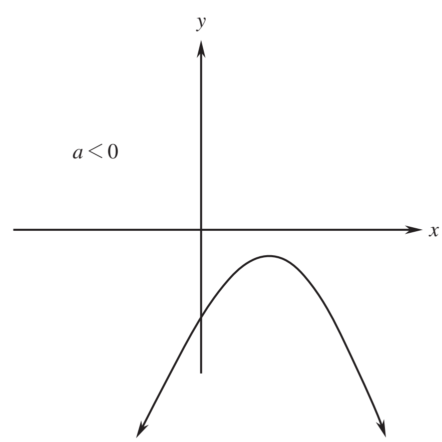
2. If $y = ax^2 + bx + c$ has 1 x-intercept, then the quadratic equation $ax^2 + bx + c = 0$ has 1 real root





3. If $y = ax^2 + bx + c$ has no x-intercepts, then the quadratic equation $ax^2 + bx + c = 0$ has no real roots





Since this graph is always positive, it is called a **positive definite** function.

Since this graph is always negative, it is called a **negative definite** function.

This information can be found without sketching the graph.

Investigation

1. Solve the following quadratic equations using the quadratic formula
   (a) $x^2 - 3x + 2 = 0$
   (b) $x^2 + 4x - 7 = 0$
   (c) $x^2 + x + 5 = 0$
   (d) $x^2 - 6x + 9 = 0$
2. Without solving a quadratic equation, can you predict how many roots it has by looking at the quadratic formula?

In the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the expression $b^2 - 4ac$ is called the **discriminant**. It gives us information about the roots of the quadratic equation $ax^2 + bx + x = 0$.

EXAMPLES

Use the quadratic formula to find how many real roots each quadratic equation has.

1. $x^2 + 5x - 3 = 0$

Solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-5 \pm \sqrt{5^2 - 4 \times 1 \times -3}}{2 \times 1}$$
$$= \frac{-5 \pm \sqrt{25 + 12}}{2}$$
$$= \frac{-5 \pm \sqrt{37}}{2}$$

There are 2 real roots:

$$x = \frac{-5 + \sqrt{37}}{2}, \frac{-5 - \sqrt{37}}{2}$$

2. $x^2 - x + 4 = 0$

Solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times 4}}{2 \times 1}$$
$$= \frac{1 \pm \sqrt{-15}}{2}$$

There are no real roots since $\sqrt{-15}$ has no real value.

3. $x^2 - 2x + 1 = 0$

Solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times 1}}{2 \times 1}$$
$$= \frac{2 \pm \sqrt{0}}{2}$$

CONTINUED

There are 2 real roots:

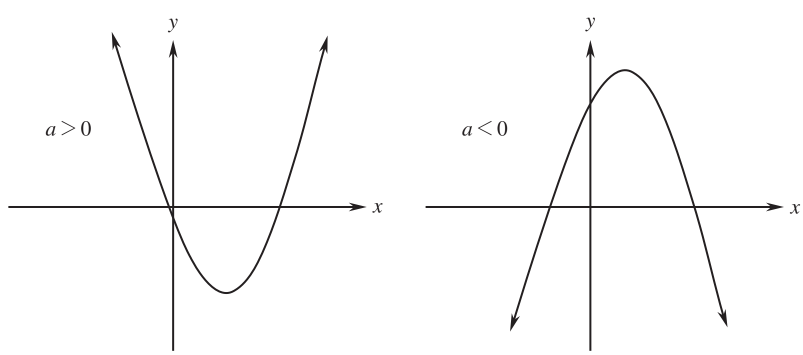
$$x = \frac{2+\sqrt{0}}{2}, \frac{2-\sqrt{0}}{2}$$
$$= 1, 1$$

However, these are equal roots.

Notice that when there are 2 real roots, the discriminant $b^2 - 4ac > 0$.
When there are 2 equal roots (or just 1 real root), $b^2 - 4ac = 0$.
When there are no real roots, $b^2 - 4ac < 0$.
We often use $\Delta = b^2 - 4ac$.

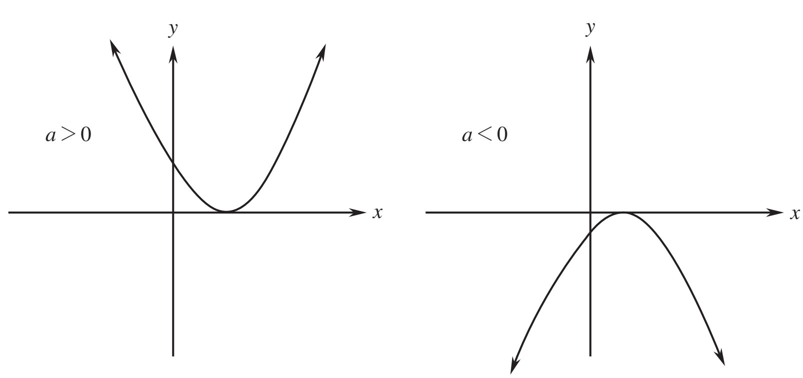
Δ is the Greek letter 'delta'.

1. If $\Delta > 0$, then the quadratic equation $ax^2 + bx + c = 0$ has 2 real unequal (different) roots.

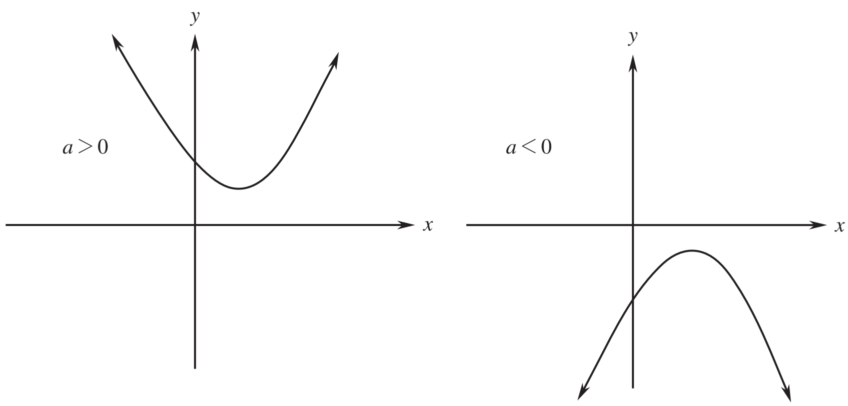


If Δ is a perfect square, the roots are rational.
If Δ is not a perfect square, the roots are irrational.

2. If $\Delta = 0$, then the quadratic equation $ax^2 + bx + c = 0$ has 1 real root or 2 equal roots.



3. If $\Delta < 0$, then the quadratic equation $ax^2 + bx + c = 0$ has no real roots.



If $\Delta < 0$ and $a > 0$, it is **positive definite** and $ax^2 + bx + c > 0$ for all x.

If $\Delta < 0$ and $a < 0$, it is **negative definite** and $ax^2 + bx + c < 0$ for all x.

We can examine the roots of the quadratic equation by using the discriminant rather than the whole quadratic formula.

EXAMPLES

1. Show that the equation $2x^2 + x + 4 = 0$ has no real roots.

Solution

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= 1^2 - 4(2)(4) \\ &= 1 - 32 \\ &= -31 \\ &< 0\end{aligned}$$

So the equation has no real roots.

2. Find the values of k for which the quadratic equation $5x^2 - 2x + k = 0$ has real roots.

Solution

For real unequal roots, $\Delta > 0$.
For real equal roots, $\Delta = 0$.
So for real roots, $\Delta \geq 0$.

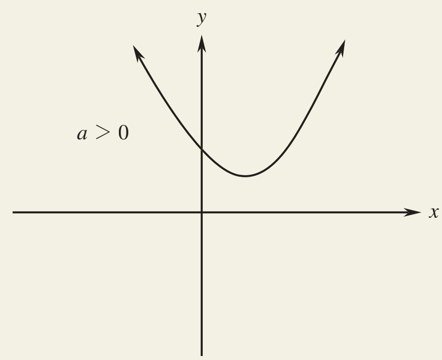
CONTINUED

$$\begin{aligned}\Delta &\geq 0\\ b^2 - 4ac &\geq 0\\ (-2)^2 - 4(5)(k) &\geq 0\\ 4 - 20k &\geq 0\\ 4 &\geq 20k\\ \frac{1}{5} &\geq k\end{aligned}$$

3. Show that $x^2 - 2x + 4 > 0$ for all x.

Solution

If $a > 0$ and $\Delta < 0$, then $ax^2 + bx + c > 0$ for all x.



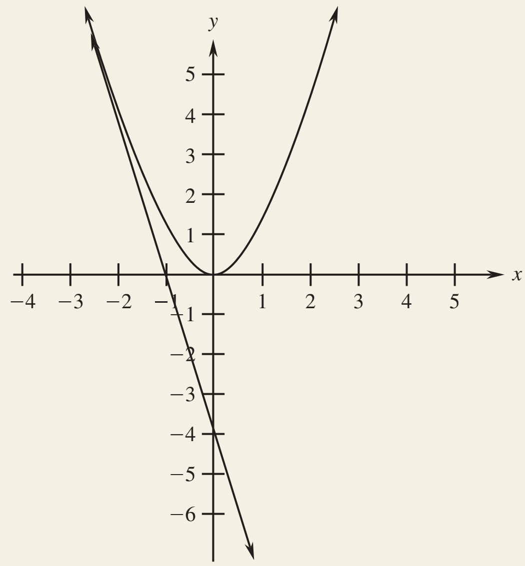
$$\begin{aligned}a &= 1\\ &> 0\\ \Delta &= b^2 - 4ac\\ &= (-2)^2 - 4(1)(4)\\ &= 4 - 16\\ &= -12\\ &< 0\end{aligned}$$

Since $a > 0$ and $\Delta < 0$, $x^2 - 2x + 4 > 0$ for all x.

4. Show that the line $4x + y + 4 = 0$ is a tangent to the parabola $y = x^2$.

Solution

For the line to be a tangent, it must intersect with the curve in only 1 point.



It is too hard to tell from the graph if the line is a tangent, so we solve simultaneous equations to find any points of intersection.

$y = x^2$ (1)

$4x + y + 4 = 0$ (2)

Substitute (1) into (2):

$4x + x^2 + 4 = 0$

$x^2 + 4x + 4 = 0$

We don't need to find the roots of the equation as the question only asks how many roots there are. We find the discriminant.

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= 4^2 - 4(1)(4) \\ &= 16 - 16 \\ &= 0 \end{aligned}$$

∴ the equation has 1 real root (equal roots) so there is only one point of intersection.

So the line is a tangent to the parabola.

10.3 Exercises

1. Find the discriminant of each quadratic equation.
   (a) $x^2 - 4x - 1 = 0$
   (b) $2x^2 + 3x + 7 = 0$
   (c) $-4x^2 + 2x - 1 = 0$
   (d) $6x^2 - x - 2 = 0$
   (e) $-x^2 - 3x = 0$
   (f) $x^2 + 4 = 0$
   (g) $x^2 - 2x + 1 = 0$
   (h) $-3x^2 - 2x + 5 = 0$
   (i) $-2x^2 + x + 2 = 0$
   (j) $-x^2 + 4x - 4 = 0$

2. Find the discriminant and state whether the roots of the quadratic equation are real or imaginary (not real), and if they are real, whether they are equal or unequal, rational or irrational.

(a) $x^2 - x - 4 = 0$
(b) $2x^2 + 3x + 6 = 0$
(c) $x^2 - 9x + 20 = 0$
(d) $x^2 + 6x + 9 = 0$
(e) $2x^2 - 5x - 1 = 0$
(f) $-x^2 + 2x - 5 = 0$
(g) $-2x^2 - 5x + 3 = 0$
(h) $-5x^2 + 2x - 6 = 0$
(i) $-x^2 + x = 0$
(j) $-2x^2 + 8x - 2 = 0$

3. Find the value of p for which the quadratic equation $x^2 + 2x + p = 0$ has equal roots.

4. Find any values of k for which the quadratic equation $x^2 + kx + 1 = 0$ has equal roots.

5. Find all the values of b for which $2x^2 + x + b + 1 = 0$ has real roots.

6. Evaluate p if $px^2 + 4x + 2 = 0$ has no real roots.

7. Find all values of k for which $(k + 2)x^2 + x - 3 = 0$ has 2 real unequal roots.

8. Prove that $3x^2 - x + 7 > 0$ for all real x.

9. Find the values of k for which $x^2 + (k + 1)x + 4 = 0$ has real roots.

10. Find values of k for which the expression $kx^2 + 3kx + 9$ is positive definite.

11. Find the values of m for which the quadratic equation $x^2 - 2mx + 9 = 0$ has real and different roots.

12. If $x^2 - 2kx + 1 = 0$ has real roots, evaluate k.

13. Find exact values of p if $px^2 - 2x + 3p = 0$ is negative definite.

14. Evaluate b if $(b - 2)x^2 - 2bx + 5b = 0$ has real roots.

15. Find values of p for which the quadratic equation $x^2 + px + p + 3 = 0$ has real roots.

16. Show that the line $y = 2x + 6$ cuts the parabola $y = x^2 + 3$ in 2 points.

17. Show that the line $3x + y - 4 = 0$ cuts the parabola $y = x^2 + 5x + 3$ in 2 points.

18. Show that the line $y = -x - 4$ does not touch the parabola $y = x^2$.

19. Show that the line $y = 5x - 2$ is a tangent to the parabola $y = x^2 + 3x - 1$.

20. The line $y = 3x - p + 1$ is a tangent to the parabola $y = x^2$. Evaluate p.

21. Which of these lines is a tangent to the circle $x^2 + y^2 = 4$?
(a) $3x - y - 1 = 0$
(b) $5x + y - 3 = 0$
(c) $4x + 3y - 10 = 0$
(d) $5x - 12y + 26 = 0$
(e) $2x + y - 7 = 0$

Quadratic Identities

When you use the quadratic formula to solve an equation, you compare a quadratic, say, $3x^2 - 2x + 5 = 0$ with the general quadratic $ax^2 + bx + c = 0$.

You are assuming when you do this that $3x^2 - 2x + 5$ and $ax^2 + bx + c$ are equivalent expressions.

We can state this as a general rule:

If two quadratic expressions are **equivalent** to each other then the corresponding **coefficients** must be **equal**.

> *If* $a_1x^2 + b_1x + c_1 \equiv a_2x^2 + b_2x + c_2$ for all real x
> then $a_1 = a_2, b_1 = b_2$ and $c_1 = c_2$

Proof

If $a_1x^2 + b_1x + c_1 = a_2x^2 + b_2x + c_2$ for more than two values of x, then
$(a_1 - a_2)x^2 + (b_1 - b_2)x + (c_1 - c_2) = 0$.
That is, $a_1 = a_2, b_1 = b_2$ and $c_1 = c_2$.

EXAMPLES

1. Write $2x^2 - 3x + 5$ in the form $A(x-1)^2 + B(x-1) + C$.

Solution

$$A(x-1)^2 + B(x-1) + C = A(x^2 - 2x + 1) + Bx - B + C$$
$$= Ax^2 - 2Ax + A + Bx - B + C$$
$$= Ax^2 + (-2A + B)x + A - B + C$$

For $2x^2 - 3x + 5 \equiv Ax^2 + (-2A + B)x + A - B + C$

$A = 2$ (1)

$-2A + B = -3$ (2)

$A - B + C = 5$ (3)

You learnt how to solve simultaneous equations with 3 unknowns in Chapter 3.

Substitute (1) into (2):

$-2(2) + B = -3$

$-4 + B = -3$

$B = 1$

Substitute $A = 2$ and $B = 1$ into (3):

$2 - 1 + C = 5$

$1 + C = 5$

$C = 4$

$\therefore 2x^2 - 3x + 5 \equiv 2(x-1)^2 + (x-1) + 4$

CONTINUED

2. Find values for a, b and c if $x^2 - x \equiv a(x+3)^2 + bx + c - 1$.

Solution

$$\begin{aligned} a(x+3)^2 + bx + c - 1 &= a(x^2 + 6x + 9) + bx + c - 1 \\ &= ax^2 + 6ax + 9a + bx + c - 1 \\ &= ax^2 + (6a+b)x + 9a + c - 1 \end{aligned}$$

For $x^2 - x \equiv ax^2 + (6a+b)x + 9a + c - 1$

$$a = 1 \quad (1)$$

$$6a + b = -1 \quad (2)$$

$$9a + c - 1 = 0 \quad (3)$$

Substitute (1) into (2):

$$\begin{aligned} 6(1) + b &= -1 \\ 6 + b &= -1 \\ b &= -7 \end{aligned}$$

Substitute (1) into (3):

$$\begin{aligned} 9(1) + c - 1 &= 0 \\ 8 + c &= 0 \\ c &= -8 \end{aligned}$$

$\therefore \quad a = 1, b = -7, c = -8$

---

3. Find the equation of the parabola that passes through the points $(-1, -3)$, $(0, 3)$ and $(2, 21)$.

Solution

The parabola has equation in the form $y = ax^2 + bx + c$. Substitute the points into the equation:

$(-1, -3)$: $\quad -3 = a(-1)^2 + b(-1) + c$

$\quad\quad = a - b + c$

$\therefore \quad a - b + c = -3 \quad (1)$

$(0, 3)$: $\quad 3 = a(0)^2 + b(0) + c$

$\quad\quad = c$

$\therefore \quad c = 3 \quad (2)$

$(2, 21)$: $\quad 21 = a(2)^2 + b(2) + c$

$\quad\quad = 4a + 2b + c$

$\therefore \quad 4a + 2b + c = 21 \quad (3)$

Solve simultaneous equations to find a, b and c.

Substitute (2) into (1):

$$a - b + 3 = -3$$

$$a - b = -6 \quad (4)$$

Substitute (2) into (3):

$$4a + 2b + 3 = 21$$

$$4a + 2b = 18 \qquad (5)$$

(4) × 2:

$$2a - 2b = -12 \qquad (6)$$

(6) + (5):

$$\begin{aligned} 2a - 2b &= -12 \\ \underline{4a + 2b} &\underline{= 18} \\ 6a \quad &= 6 \\ a &= 1 \end{aligned}$$

Substitute $a = 1$ into (5):

$$\begin{aligned} 4(1) + 2b &= 18 \\ 4 + 2b &= 18 \\ 2b &= 14 \\ b &= 7 \end{aligned}$$

$\therefore \quad a = 1, b = 7, c = 3$

Thus the parabola has equation $y = x^2 + 7x + 3$.

10.4 Exercises

1. Find values of a, b and c for which
   (a) $x^2 + 4x - 3 \equiv a(x + 1)^2 + b(x + 1) + c$
   (b) $2x^2 - 3x + 1 \equiv a(x + 2)^2 + b(x + 2) + c$
   (c) $x^2 - x - 2 \equiv a(x - 1)^2 + b(x - 1) + c$
   (d) $x^2 + x + 6 \equiv a(x - 3)^2 + b(x - 3) + c$
   (e) $3x^2 - 5x - 2 \equiv a(x + 1)^2 + b(x - 1) + c$
   (f) $4x^2 + x - 7 \equiv a(x - 2)^2 + b(x - 2) + c$
   (g) $2x^2 + 4x - 1 \equiv a(x + 4)^2 + b(x + 2) + c$
   (h) $3x^2 - 2x + 5 \equiv a(x + 1)^2 + bx + c$
   (i) $-x^2 + 4x - 3 \equiv a(x + 3)^2 + b(x + 3) + c$
   (j) $-2x^2 + 4x - 3 \equiv a(x - 1)^2 + b(x + 1) + c$

2. Find values of m, p and q for which $2x^2 - x - 1 \equiv m(x + 1)^2 + p(x + 1) + q$.

3. Express $x^2 - 4x + 5$ in the form $Ax(x - 2) + B(x + 1) + C + 4$.

4. Show that $x^2 + 2x + 9$ can be written in the form $a(x - 2)(x + 3) + b(x - 2) + c$ where $a = 1$, $b = 1$ and $c = 17$.

5. Find values of A, B and C if $x^2 + x - 2 \equiv A(x - 2)^2 + Bx + C$.

6. Find values of a, b and c for which $3x^2 + 5x - 1 \equiv ax(x + 3) + bx^2 + c(x + 1)$.

7. Evaluate K, L and M if $x^2 \equiv K(x - 3)^2 + L(x + 1) - 2M$.

8. Express $4x^2 + 2$ in the form $a(x + 5) + b(2x - 3)^2 + c - 2$.

9. Find the values of a, b and c if $20x - 17 \equiv a(x - 4)^2 - b(5x + 1) + c$.

10. Find the equation of the parabola that passes through the points
    (a) $(0, -5)$, $(2, -3)$ and $(-3, 7)$
    (b) $(1, -2)$, $(3, 0)$ and $(-2, 10)$
    (c) $(-2, 21)$, $(1, 6)$ and $(-1, 12)$
    (d) $(2, 3)$, $(1, -4)$ and $(-1, -12)$
    (e) $(0, 1)$, $(-2, 1)$ and $(2, -7)$

Sum and Product of Roots

When you solve a quadratic equation, you may notice a relationship between the roots. You also used this to factorise trinomials in Chapter 2.

EXAMPLE

(a) Solve $x^2 - 9x + 20 = 0$.
(b) Find the sum of the roots.
(c) Find the product of the roots.

Solution

(a)
$$x^2 - 9x + 20 = 0$$
$$(x - 4)(x - 5) = 0$$
$$x - 4 = 0, x - 5 = 0$$
$$\therefore x = 4, \qquad x = 5$$

(b)
$$\text{Sum} = 4 + 5 = 9$$

Notice −9 is the coefficient of x and 20 is the constant term in the equation.

(c)
$$\text{Product} = 4 \times 5 = 20$$

This relationship with the sum and product of the roots works for any quadratic equation.

> The general quadratic equation can be written in the form
> $$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$
> where α and β are the roots of the equation.

Proof

Suppose the general quadratic equation $ax^2 + bx + c = 0$ has roots α and β. Then this equation can be written in the form

$$(x-\alpha)(x-\beta)=0$$
$$\text{i.e. } x^2-\beta x-\alpha x+\alpha\beta=0$$
$$x^2-(\alpha+\beta)x+\alpha\beta=0$$

EXAMPLES

1. Find the quadratic equation that has roots 6 and -1.

Solution

Method 1: Using the general formula

$x^2-(\alpha+\beta)x+\alpha\beta=0$ where $\alpha=6$ and $\beta=-1$

It doesn't matter which way around we name these roots.

$$\begin{aligned}\alpha+\beta &= 6+-1\\ &= 5\\ \alpha\beta &= 6\times-1\\ &= -6\end{aligned}$$

Substituting into $x^2-(\alpha+\beta)x+\alpha\beta=0$ gives

$x^2-5x-6=0$

Method 2:

If 6 and -1 are the roots of the equation then it can be written as

$$(x-6)(x+1)=0$$
$$x^2+x-6x-6=0$$
$$x^2-5x-6=0$$

---

2. Find the quadratic equation that has roots $3+\sqrt{2}$ and $3-\sqrt{2}$.

Solution

Method 1: Using the general formula

$$\begin{aligned}\alpha+\beta &= 3+\sqrt{2}+3-\sqrt{2}\\ &= 6\\ \alpha\beta &= (3+\sqrt{2})\times(3-\sqrt{2})\\ &= 3^2-(\sqrt{2})^2\\ &= 9-2\\ &= 7\end{aligned}$$

Substituting into $x^2-(\alpha+\beta)x+\alpha\beta=0$ gives

$x^2-6x+7=0$

Method 2:

If $3+\sqrt{2}$ and $3-\sqrt{2}$ are the roots of the equation then it can be written as

$$(x-\{3+\sqrt{2}\})(x-\{3-\sqrt{2}\})=0$$
$$(x-3-\sqrt{2})(x-3+\sqrt{2})=0$$
$$x^2-3x+\sqrt{2}x-3x+9-3\sqrt{2}-\sqrt{2}x+3\sqrt{2}-2=0$$
$$x^2-6x+7=0$$

We can find a more general relationship between the sum and product of roots of a quadratic equation.

> If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$:
>
> Sum of roots: $\alpha + \beta = -\dfrac{b}{a}$
>
> Product of roots: $\alpha\beta = \dfrac{c}{a}$

Proof

If an equation has roots α and β, it can be written as $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.

But we know that α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$.

Using quadratic identities, we can compare the two forms of the equation.

$$ax^2 + bx + c = 0$$

$$\frac{ax^2}{a} + \frac{bx}{a} + \frac{c}{a} = \frac{0}{a}$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

For $x^2 - (\alpha + \beta)x + \alpha\beta \equiv x^2 + \dfrac{b}{a}x + \dfrac{c}{a}$

$$-(\alpha + \beta) = \frac{b}{a}$$

$\therefore$ $\qquad \alpha + \beta = -\dfrac{b}{a}$

Also $\qquad \alpha\beta = \dfrac{c}{a}$

EXAMPLES

1. Find (a) $\alpha + \beta$ (b) $\alpha\beta$ (c) $\alpha^2 + \beta^2$ if α and β are the roots of $2x^2 - 6x + 1 = 0$.

Solution

(a) $\alpha + \beta = -\dfrac{b}{a}$

$= -\dfrac{(-6)}{2}$

$= 3$

(b) $\alpha\beta = \frac{c}{a}$

$= \frac{1}{2}$

(c)
$$\alpha^2 + \beta^2 \neq (\alpha + \beta)^2$$
$$(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$$
$$(\alpha + \beta)^2 - 2\alpha\beta = \alpha^2 + \beta^2$$
$$(3)^2 - 2\left(\frac{1}{2}\right) = \alpha^2 + \beta^2$$
$$9 - 1 = \alpha^2 + \beta^2$$
$$8 = \alpha^2 + \beta^2$$

2. Find the value of k if one root of $kx^2 - 7x + k + 1 = 0$ is -2.

Solution

If -2 is a root of the equation then $x = -2$ satisfies the equation.
Substitute $x = -2$ into the equation:

$$k(-2)^2 - 7(-2) + k + 1 = 0$$
$$4k + 14 + k + 1 = 0$$
$$5k + 15 = 0$$
$$5k = -15$$
$$k = -3$$

3. Evaluate p if one root of $x^2 + 2x - 5p = 0$ is double the other root.

Solution

You could use β and 2β instead.

If one root is α then the other root is 2α.
Sum of roots:

$$\alpha + \beta = -\frac{b}{a}$$
$$\alpha + 2\alpha = -\frac{2}{1}$$
$$3\alpha = -2$$
$$\alpha = -\frac{2}{3}$$

CONTINUED

Product of roots:

$$\alpha\beta = \frac{c}{a}$$

$$\alpha \times 2\alpha = \frac{-5p}{1}$$

$$2\alpha^2 = -5p$$

$$2\left(-\frac{2}{3}\right)^2 = -5p$$

$$2\left(\frac{4}{9}\right) = -5p$$

$$\frac{8}{9} = -5p$$

$$-\frac{8}{45} = p$$

10.5 Exercises

1. Find $\alpha + \beta$ and $\alpha\beta$ if α and β are the roots of
   (a) $x^2 + 2x + 1 = 0$
   (b) $2x^2 - 3x - 6 = 0$
   (c) $5x^2 - x - 9 = 0$
   (d) $x^2 + 7x + 1 = 0$
   (e) $3y^2 - 8y + 3 = 0$

2. If α and β are the roots of the quadratic equation $x^2 - 3x - 6 = 0$, find the value of
   (a) $\alpha + \beta$
   (b) $\alpha\beta$
   (c) $\frac{1}{\alpha} + \frac{1}{\beta}$
   (d) $\alpha^2 + \beta^2$

3. Find the quadratic equation whose roots are
   (a) 2 and -5
   (b) -3 and 7
   (c) -1 and -4
   (d) $4 + \sqrt{5}$ and $4 - \sqrt{5}$
   (e) $1 + 2\sqrt{7}$ and $1 - 2\sqrt{7}$

4. Find the value of m in $x^2 + 2mx - 6 = 0$ if one of the roots is 2.

5. If one of the roots of the quadratic equation $2x^2 - 5x + k - 1 = 0$ is -3, find the value of k.

6. One root of $3x^2 - 2(3b + 1)x + 4b = 0$ is 8. Find the value of b.

7. In the quadratic equation $2x^2 - 3x + k = 0$, one root is double the other. Find the value of k.

8. In the quadratic equation $x^2 - 8x + p - 1 = 0$, one root is triple the other. Find the value of p.

9. In the quadratic equation $(k - 2)x^2 + 50x + 2k + 3 = 0$, the roots are reciprocals of each other. Find the value of k.

Reciprocals are n and $\frac{1}{n}$.

10. In the quadratic equation $x^2 + mx + 2 = 0$, the roots are consecutive. Find the values of m.

Consecutive numbers are numbers that follow each other in order, such as 3 and 4.

11. In the quadratic equation $-3x^2 - (k+1)x + 5 = 0$, the roots are equal in magnitude but opposite in sign. Find the value of k.

12. Find values of n in the equation $2x^2 - 5(n-1)x + 12 = 0$ if the two roots are consecutive.

13. If the sum of the roots of $x^2 + px + r = 0$ is -2 and the product of roots is -7, find the values of p and r.

14. One root of the quadratic equation $x^2 + bx + c = 0$ is 4 and the product of the roots is 8. Find the values of b and c.

15. The roots of the quadratic equation $x^2 + 4x - a = 0$ are $b + 1$ and $b - 3$. Find the values of a and b.

16. Show that the roots of the quadratic equation $3mx^2 + 2x + 3m = 0$ are always reciprocals of one another.

17. Find values of k in the equation $x^2 + (k+1)x + \left(\frac{k+1}{4}\right) = 0$ if:
(a) roots are equal in magnitude but opposite in sign
(b) roots are equal
(c) one root is 1
(d) roots are reciprocals of one another
(e) roots are real.

18. Find exact values of p in the equation $x^2 + px + 3 = 0$ if
(a) the roots are equal
(b) it has real roots
(c) one root is double the other.

19. Find values of k in the equation $x^2 + kx + k - 1 = 0$ if
(a) the roots are equal
(b) one root is 4
(c) the roots are reciprocals of one another.

20. Find values of m in the equation $mx^2 + x + m - 3 = 0$ if
(a) one root is -2
(b) it has no real roots
(c) the product of the roots is 2.

Equations Reducible to Quadratics

To solve a quadratic equation such as $(x-3)^2 - (x-3) - 2 = 0$, you could expand the brackets and then solve the equation. However, in this section you will learn a different way to solve this.

There are other equations that do not look like quadratic equations that can also be solved this way.

EXAMPLES

1. Solve $(x + 2)^2 - 3(x + 2) - 4 = 0$.

Solution

Let $u = x + 2$

Then $u^2 - 3u - 4 = 0$

$(u - 4)(u + 1) = 0$

$u - 4 = 0, \quad u + 1 = 0$

$u = 4, \quad u = -1$

But $u = x + 2$

So $x + 2 = 4, \quad x + 2 = -1$

$x = 2, \quad x = -3$

---

2. Solve $x + \frac{2}{x} = 3$ where $x \neq 0$.

Solution

$$x + \frac{2}{x} = 3$$

$$x \times x + \frac{2}{x} \times x = 3 \times x$$

$$x^2 + 2 = 3x$$

$$x^2 - 3x + 2 = 0$$

$$(x - 2)(x - 1) = 0$$

$x - 2 = 0, \quad x - 1 = 0$

$x = 2, \quad x = 1$

---

3. Solve $9^x - 4.3^x + 3 = 0$.

Solution

$9^x = (3^2)^x = (3^x)^2$

So $9^x - 4.3^x + 3 = 0$ can be written as

$(3^x)^2 - 4.3^x + 3 = 0$

Let $k = 3^x$

$k^2 - 4k + 3 = 0$

$(k - 3)(k - 1) = 0$

$k - 3 = 0, \quad k - 1 = 0$

$k = 1, \quad k = 3$

But $k = 3^x$

So $3^x = 1, \quad 3^x = 3$

$x = 0, \quad x = 1$

4. Solve $2\sin^2 x + \sin x - 1 = 0$ for $0° \le x \le 360°$.

Solution

Let $\sin x = u$

Then $2u^2 + u - 1 = 0$

$(2u - 1)(u + 1) = 0$

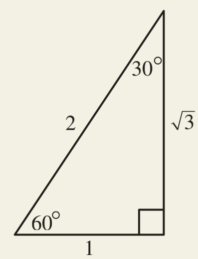
$2u - 1 = 0$ or $u + 1 = 0$

$2u = 1 \qquad u = -1$

$u = \frac{1}{2}$

But $u = \sin x$

So $\sin x = \frac{1}{2}$ or $\sin x = -1$



$\sin x = \frac{1}{2}$ has solutions in the 1st and 2nd quadrants

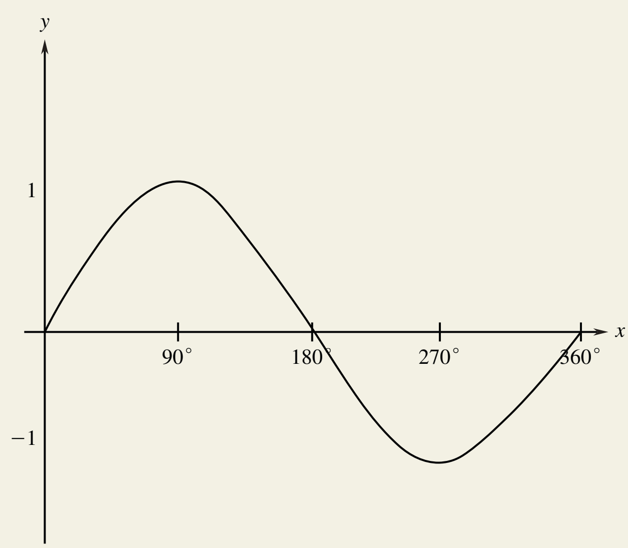
$\sin 30° = \frac{1}{2}$

See Chapter 6 if you have forgotten how to solve a trigonometric equation.

So $x = 30°, 180° - 30°$

$= 30°, 150°$

For $\sin x = -1$, we use the graph of $y = \sin x$



From the graph:

$x = 270°$

So solutions to $2\sin^2 x + \sin x - 1 = 0$ are

$x = 30°, 150°, 270°$

CONTINUED

10.6 Exercises

1. Solve
   (a) $(x-1)^2 + 7(x-1) + 10 = 0$
   (b) $(y-3)^2 - (y-3) - 2 = 0$
   (c) $(x+2)^2 - 2(x+2) - 8 = 0$
   (d) $(n-5)^2 + 7(n-5) + 6 = 0$
   (e) $(a-4)^2 + 6(a-4) - 7 = 0$
   (f) $(p+1)^2 - 9(p+1) + 20 = 0$
   (g) $(x+3)^2 - 4(x+3) - 5 = 0$
   (h) $(k-8)^2 - (k-8) - 12 = 0$
   (i) $(t-2)^2 + 2(t-2) - 24 = 0$
   (j) $(b+9)^2 - 2(b+9) - 15 = 0$

2. Solve $(x \neq 0)$.
   (a) $x - \frac{6}{x} = 1$
   (b) $x + \frac{6}{x} = 5$
   (c) $x + \frac{20}{x} - 9 = 0$
   (d) $x + \frac{15}{x} = 8$
   (e) $2x + \frac{12}{x} = 11$

3. Solve
   (a) $x^4 - 7x^2 - 18 = 0$
   (b) $y^4 - 6y^2 + 8 = 0$, giving exact values
   (c) $(x^2 - x)^2 + (x^2 - x) - 2 = 0$ giving exact values
   (d) $(x^2+3x-1)^2 - 7(x^2+3x-1)+10=0$ correct to 2 decimal places
   (e) $(a^2 + 4a)^2 + 2(a^2 + 4a) - 8 = 0$ giving exact values.

4. Solve
   (a) $2^{2x} - 9.2^x + 8 = 0$
   (b) $3^{2p} + 3^p - 12 = 0$
   (c) $5^{2x} - 5^x - 20 = 0$
   (d) $9^x + 3^x - 12 = 0$
   (e) $4^x - 10.2^x + 16 = 0$

5. Solve $x^2 + \frac{4}{x^2} = 5\ (x \neq 0)$.

6. Solve $\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) - 2 = 0$ $(x \neq 0)$.

7. Solve $\left(x^2 + \frac{1}{x^2}\right)^2 - 9\left(x^2 + \frac{1}{x^2}\right) + 20 = 0$ correct to 2 decimal places $(x \neq 0)$.

8. Solve for $0° \leq x \leq 360°$.
   (a) $\sin^2 x - \sin x = 0$
   (b) $\cos^2 x + \cos x = 0$
   (c) $2\sin^2 x - \sin x - 1 = 0$
   (d) $2\cos^2 x = \cos x$
   (e) $\sin x = \cos^2 x - 1$

9. Solve for $0° \leq x \leq 360°$.
   (a) $\tan^2 x - \tan x = 0$
   (b) $\cos^2 x - 1 = 0$
   (c) $2\sin^2 x - \sin x = 0$
   (d) $8\sin^4 x - 10\sin^2 x + 3 = 0$
   (e) $3\tan^4 x - 10\tan^2 x + 3 = 0$

10. Show that the equation $x + 3 + \frac{2}{x+3} = 5$ has 2 real irrational roots $(x \neq -3)$.

Test Yourself 10

1. Solve
(a) $x^2 - 3x \le 0$
(b) $n^2 - 9 > 0$
(c) $4 - y^2 \ge 0$

2. Evaluate a, b and c if $2x^2 - 5x + 7 = 2a(x + 1)^2 + b(x + 1) + c$.

3. Find
(a) the equation of the axis of symmetry and
(b) the minimum value of the parabola $y = x^2 - 4x + 1$.

4. Show that $y = x^2 - 2x + 7$ is a positive definite quadratic function.

5. If α and β are roots of the quadratic equation $x^2 - 6x + 3 = 0$, find
(a) $\alpha + \beta$
(b) $\alpha\beta$
(c) $\frac{1}{\alpha} + \frac{1}{\beta}$
(d) $\alpha\beta^2 + \alpha^2\beta$
(e) $\alpha^2 + \beta^2$

6. Solve $(3x - 2)^2 - 2(3x - 2) - 3 = 0$.

7. Describe the roots of each quadratic equation as
(i) real, different and rational
(ii) real, different and irrational
(iii) equal or
(iv) unreal.
   (a) $2x^2 - x + 3 = 0$
   (b) $x^2 - 10x - 25 = 0$
   (c) $x^2 - 10x + 25 = 0$
   (d) $3x^2 + 7x - 2 = 0$
   (e) $6x^2 - x - 2 = 0$

8. Show that $-4 + 3x - x^2 < 0$ for all x.

9. Find
(a) the equation of the axis of symmetry and
(b) the maximum value of the quadratic function $y = -2x^2 - x + 6$.

10. Write $3x^2 + 7$ in the form $a(x - 2)^2 + b(x + 3) + c$.

11. Solve $2 \sin^2 x + \sin x - 1 = 0$ for $0° \le x \le 360°$.

12. Find the value of k in $x^2 + 3x + k - 1 = 0$ if the quadratic equation has
(a) equal roots
(b) one root -3
(c) one root double the other
(d) consecutive roots
(e) reciprocal roots.

13. Solve $2x = 5 + \frac{3}{x} \quad (x \ne 0)$.

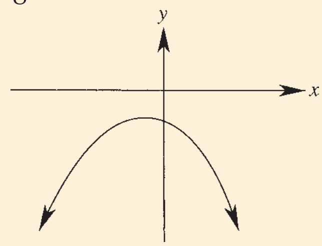
14. Find values of m such that $mx^2 + 3x - 4 < 0$ for all x.

15. Solve $5^{2x} - 26.5^x + 25 = 0$.

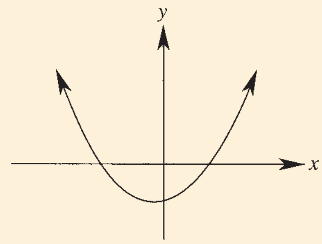
16. For each set of graphs, state whether they have
(i) 2 points
(ii) 1 point
(iii) no points of intersection.
   (a) $xy = 7$ and $3x - 5y - 1 = 0$
   (b) $x^2 + y^2 = 9$ and $y = 3x - 3$
   (c) $x^2 + y^2 = 1$ and $x - 2y - 3 = 0$
   (d) $y = \frac{2}{x}$ and $y = 3x + 1$
   (e) $y = x^2$ and $y = 4x - 4$

17. State if each quadratic function is
(i) indefinite
(ii) positive definite or
(iii) negative definite.

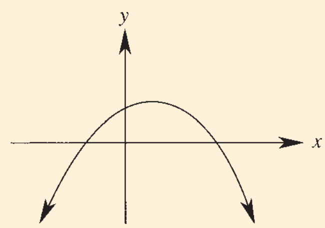
(a)



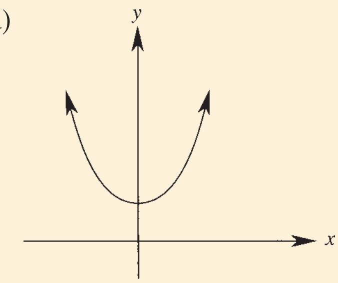
(b)



(c)



(d)



18. Show that $kx^2 - px + k = 0$ has reciprocal roots for all x.

19. Find the quadratic equation that has roots
(a) 4 and -7
(b) $5 + \sqrt{7}$ and $5 - \sqrt{7}$

20. Solve $2^{2x} - 10.2^x + 16 = 0$.

21. Solve
(a) $\dfrac{3}{x+1} < 7$
(b) $\dfrac{2n}{n-3} \geq 1$
(c) $\dfrac{2}{5y-1} > 3$
(d) $\dfrac{3x}{2x+5} \leq 2$
(e) $\dfrac{2x+1}{x-4} \geq 5$

Challenge Exercise 10

1. Show that the quadratic equation $2x^2 - kx + k - 2 = 0$ has real rational roots.

2. Find the equation of a quadratic function that passes through the points $(-2, 18), (3, -2)$ and $(1, 0)$.

3. Find the value of a, b and c if $x^2 + 5x - 3 \equiv ax(x+1) + b(x+1)^2 + cx$.

4. Solve $x^2 + 1 + \dfrac{25}{x^2+1} = 10$.

5. Find the maximum value of the function $f(x) = -2x^2 - 4x + 9$.

6. Find the value of n for which the equation $(n+2)x^2 + 3x - 5 = 0$ has one root triple the other.

7. Find the values of p for which $x^2 - x + 3p - 2 > 0$ for all x.

8. Show that the quadratic equation $x^2 - 2px + p^2 = 0$ has equal roots.

9. Solve $2^{2x+1} - 5.2^x + 2 = 0$.

10. Find values of A, B and C if $4x^2 - 3x + 7 \equiv (Ax + 4)^2 + B(x + 4) + C$.

11. Express $\dfrac{4x + 1}{x^2 - x - 2}$ in the form $\dfrac{a}{x - 2} + \dfrac{b}{x + 1}$.

12. Find exact values of k for which $x^2 + 2kx + k + 5 = 0$ has real roots.

13. Solve $3 - 2\cos^2 x - 3\sin x = 0$ for $0° \leq x \leq 360°$.

14. Solve $\left(x + \dfrac{1}{x}\right)^2 - 5\left(x + \dfrac{1}{x}\right) + 6 = 0$.

15. Solve $2\sin^2 x + \cos x - 2 = 0$ for $0° \leq x \leq 360°$.

16. If α and β are the roots of the quadratic equation $2x^2 + 4x - 5 = 0$, evaluate $\alpha^3 + \beta^3$.